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Geometry and Physics: A Didactic Proposal

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"The scientist does not study nature because it is useful to do so. He studies it because he takes pleasure in it, and he takes pleasure in it because it is beautiful. If nature were not beautiful it would not be worth knowing, and life would not be worth living. I am not speaking, of course, of the beauty which strikes the senses, of the beauty of qualities and appearances. I am far from despising this, but it has nothing to do with science. What I mean is that more intimate beauty which comes from the harmonious order of its parts, and which a pure intelligence can grasp."

Henri Poincaré - The selection of facts

Introduction and Acknowledgements

The aim of the present work is to help teachers who plan to introduce the theory of relativity to their students. Indeed, nowadays teachers are asked to update the school curriculum and treat topics of modern physics like quantum mechanics and general relativity [1]. This is indispensable since the school has to give students the tools to understand the society in which they live and the technology they use, as is argued in [2]. On the other hand, without a daring attempt to revise physics and mathematics curricula, these topics run the risk to be conveyed only as a popular subject with no true insight. Indeed, while Newtonian mechanics, thermodynamics and electromagnetism, together with other classical topics of physics, are meaningfully treated with elementary mathematics. It is matter of recent educational physics research how to teach more recent theories of physics.
Many works in literature focus on the students difficulties and misconceptions, this work focuses on how to help students overcome their difficulties.

We want to preliminarly stress that, in our opinion, it should be avoided to introduce features of modern physics starting from popular aphorisms of great scientists or from often obscure epistemelogical debates among them. This is especially true when we think of quantum mechanics, where the original disputes among scientists like Bohr, Einstein and Heisenberg cannot be understood without a basic understanding of the physical theory.

Searching a way to introduce in a meaningful way the theory of relativity to students of high school, I participated to training meeting for teachers and I designed laboratorial activities experimented in classrooms. These activities are described in the present work. I have presented just a few ideas, centred on special relativity and non-Euclidean geometry, but during the meeting the discussion was centered on how to revise mathematics and physics curriculum. On one side the need for experimental activity, which should form the core of the physics curriculum, has been coped taking advantage of the wide material produced in the *LES Project* (Laboratori per l'Educazione alla Scienza) that can be found on http://www.les.unina.it/. Teachers have tried some of the experience described, adapting it to the needs of their classrooms. The activities were documented and they were discussed in the subsequent meetings.
On the other side there is need for a formalization of the advanced concepts. The idea is to start from computations and calculations that students can perform through and through. A starting point is the discretization of differential equations. The wave equation, or even Schrödinger equation, can be solved numerically in one dimension also

with elementary mathematics, as shown in [3]. With the help of modern computers and instruments like the spreadsheets, students can gain astonishing insight into physical systems through numerical methods. In this work we also propose to approach the three-body problems, computing the trajectory of two planets that turns around the Sun, with numerical methods. Anyway, I have chosen to use *Python* programming language (see appendix B) to implement the algorithms. I believe it is important for students to approach coding technologies like Python useful for teachers and students to illustrate physics concepts and develop a sense of physical intuition through simulations and modelling.

The pedagogical framework in which the activities are designed are the ones of Vygotskij and Sfard. The *expert* must put himself in the zone of proximal development, proposing things that students does not know, but can actually understand. Furthermore, the class work is a collective study where the interactions between pairs and with the expert are the true propelling factor [4]. On the other hand, mathematical concepts are fist introduced as operation to perform on concrete objects. A paradigmatic example is the division of natural numbers, where the activities aims to catalyze the cycle of internalization, condensation and reification, until students perceives the operation as a mathematical object, the positive rational numbers [5].
The methodologies we propose are also based *...on a rich and growing body of research on teaching and learning in science, as well as on nearly two decades of efforts to define foundational knowledge and skills for K-12 science and engineering ... We focus on a limited number of disciplinary core ideas and crosscutting concepts, designed so that students continually build on and revise their knowledge and abilities over multiple years, and support the integration of such knowledge and abilities with the practices needed to engage in scientific inquiry ...* [6].

The thesis is structured miming a textbook. Topics are presented starting from the simplest ones, where only the knowledge of physics and mathematics generally owned by students in high-school is assumed and later proceeding to more complex topics like general relativity. Every topic is introduced with exercises and laboratorial activity but most of the interesting aspect of rigorous mathematics are not discussed for the sake of brevity. Nevertheless, the work has been carried out having in mind modularity criteria so that a teacher may chose any set of topics, if any, to propose in classroom.
In the first chapter a way to introduce Galilean transformations is proposed. Distance and angles are defined and particular relevance is given to the operations of translation and rotation of the system of reference. Successively, it follows a description of the *pentalaser*, a didactic tools that allows various experiments to be performed in classrooms. In

the last section, time and motion are introduced and the concept of relative motion is discussed with the aid of a motion sensor.
In the second chapter a way to introduce special relativity is proposed. In the first section Maxwell equations are presented and the wave equation is derived. Afterward, it follows a description of the microwave optics bench and the Michelson interferometer with an account of classroom activity in which various experiment have been presented. Later, the Lorentz transformation is presented as a rotation in space-time and the special theory of relativity is introduced with geometric concepts and thought experiments. In the last section, relativistic mechanics is discussed with the introduction of the four-vectors.
In the third chapter, a way to approach the general theory of relativity is introduced. In the first section, non-Euclidean geometry is presented trough measurement of angles, lengths and areas on a sphere. After that, tensors are introduced and the Einstein equation is discussed with some solutions. In the last section numerical simulations are implemented to show how astronomical predictions differs in the different theories.

This work would not have been possible without the teachers that hosted me in their classrooms, trusting my ideas. I want to thank *Maria Rosaria Camarda*, *Annette Luongo* and *Maria Loffredo* from the "Liceo Scientifico Statale Carlo Urbani", *Margherita D'Urzo* and *Ilaria Limoncelli* from the "Liceo Scientifico Statale Filippo Silvestri" and *Marina De Cesare* and *Chiara Tarallo* from the "Liceo Scientifico Statale Calamandrei. It seems that no student has been phisically or psychologically harmed as a consequence of this thesis work.

I want to thank my teachers, who never gave up on me, because they were not only *transmitters of knowledge*, but life-examples to follow. Among them *Giorgio Montalto*, who inspired my passion for theoretical aspects of physics and *Orietta Laurenza*, who inspired the habit of inquiry, *Marina Cerza*, who never stopped feeding my curiosity and *Rosalba Cerbone*, who believed in me even in the hardest times. I also want to thank my supervisors, *Emilio Balzano*, *Rodolfo Figari* and *Ofelia Pisanti* for their patience and their guidance. Furthermore, I want to thank professor *Antonio Sasso*, whose experience in teaching optics has been a beacon to follow for me.
I also wish to thank *Giuseppe Bausilio* and the *Ing. Gaetano Mascetta* for their unconditioned friendship among all these years.
And I want to thank my family, who always supported and "sopported" me (sorry for the italian slang).
And above all, I want to thank *Sarah̃* for loving and caring me.

Chapter 1

Time and Relative Dimension in Space

> The task of the educator is to make the child's spirit pass again where its forefathers have gone, moving rapidly through certain stages but suppressing none of them. In this regard, the history of science must be our guide.
>
> Henri Poincaré - L'enseignement mathématique

The goal of this chapter is to present Galilean transformations and classical mechanics in a way easily generalizable to Lorentz transformation and relativistic mechanics. In the first two sections distance, angles and reference system are introduced with the help of exercises, drawings and laboratorial activities with the penta-laser. After that, time is introduced as an astronomical measure. The concept of relative motion is introduced through a laboratorial activity with the motion sensors and Galilean transformations are finally discussed trough exercises. In the last section classical mechanics is presented with its symmetries with respect to the Galilean transformation and numerical simulations are performed to show the predictive power of the theory.

1.1 Distance and reference systems

> It is by logic that we prove, but by intuition that we discover. To know how to criticize is good, to know how to create is better.
>
> Henri Poincaré - Mathematical Definitions and Education

An exercise that could be proposed to students is to measure the distance between two points without tilting the rule with respect to the vertical and horizontal directions. Hopefully, some students will propose to build a right-angled triangle where the hypotenuse is the desired distance, as shown in 1.1.

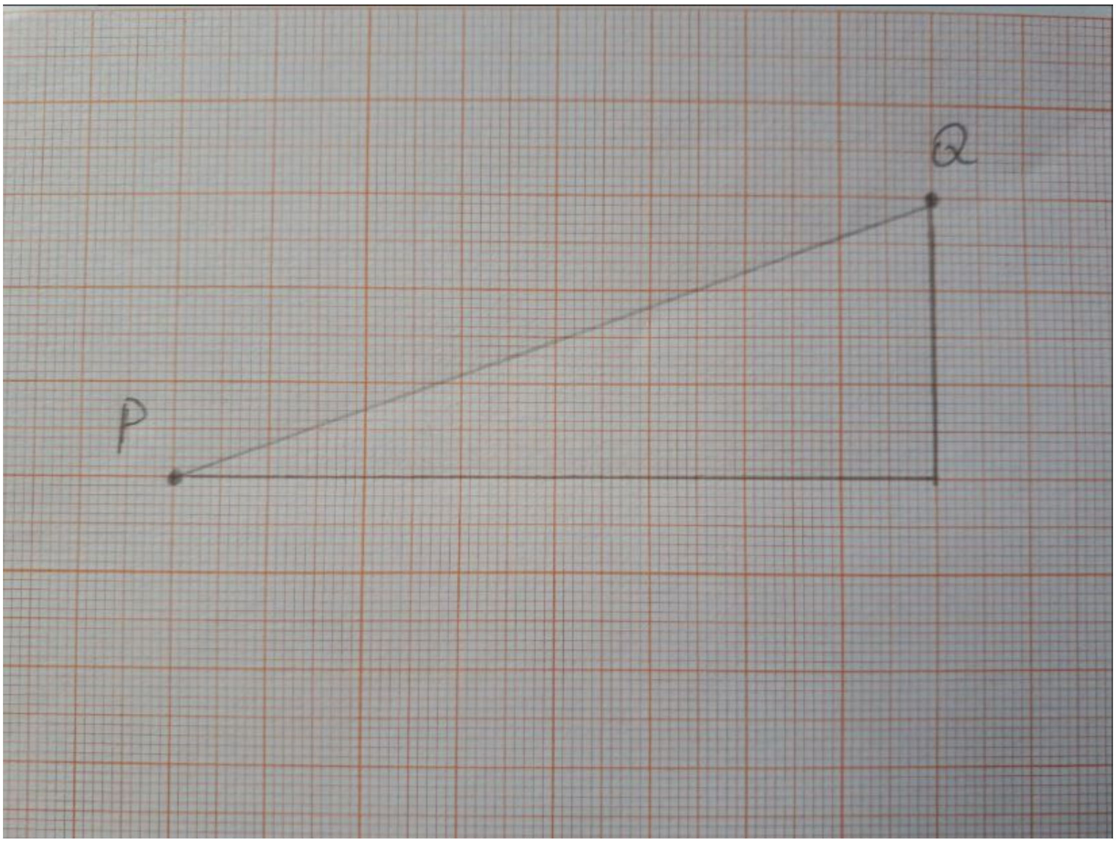


FIGURE 1.1: A right-angled triangle

In this exercise the *distance* is the thing that is measured with a rule. Later on, further activities shall be proposed that will precise the definition, giving a more insightful idea of distance. Until then, we shall focus on the idea of measuring a distance indirectly from the measures of other two distances as a pure thought-play. We can give the students two points on a Cartesian coordinate system *without* giving the measure units as shown in 1.2.

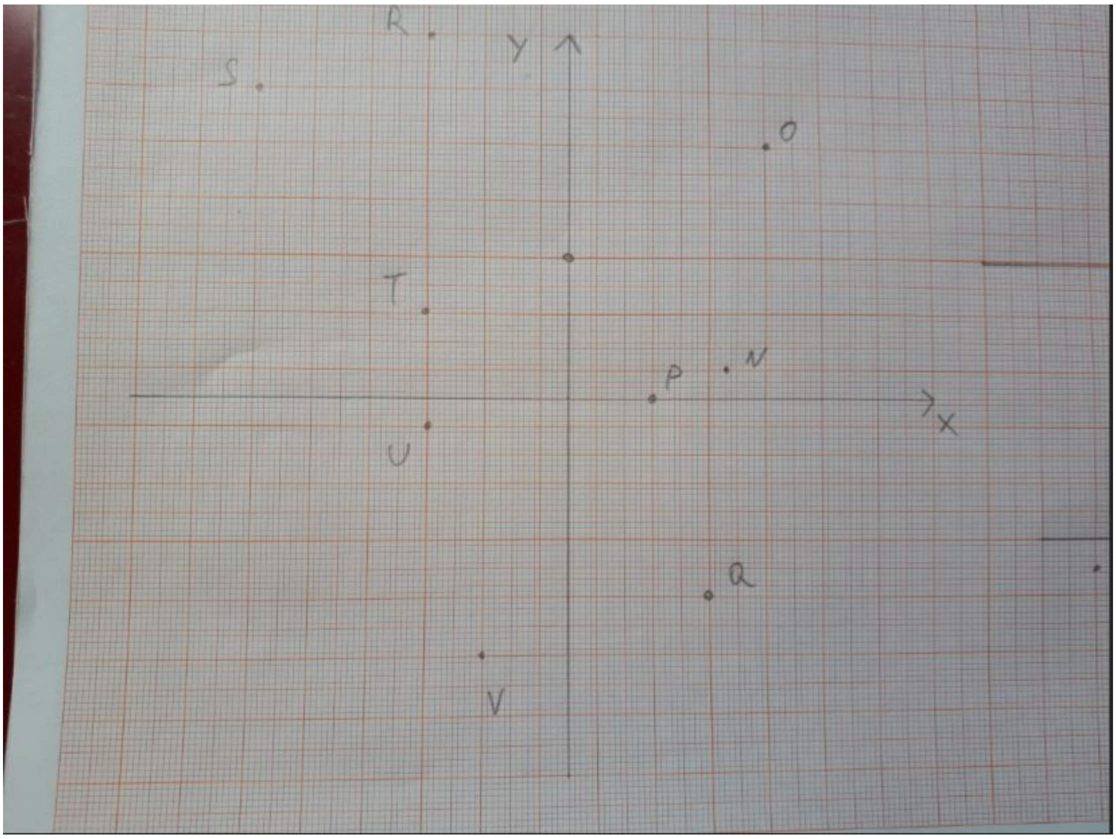


FIGURE 1.2: Cartesian coordinate systems where the students have to choose the unit of measure for the axes.

and ask the students to measure the distance between a couple of points (without tilting the rule). This way students are forced to decide a unit of measure for each axis, record the coordinates of each point and then to compute the distance with the Pythagorean theorem. This activity has two goals: on one side to recall to students the basic notion of analytic geometry they should already have, on the other side it poses the Pythagorean theorem under a new light, it can be used to give an algebraic definition of distance in two or more dimensions. Students should be led to notice that if two or more points are in line, we can assign to them *positions* as the distance from a starting point as shown in figure 1.3, and an algebraic definition of the distance of two points is, in that case, just the subtraction of the two position.

If the points are not in line, we can do the same but we need a Cartesian plane and every position is given by two numbers, not only one, and the distance becomes

$$d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1.1}$$

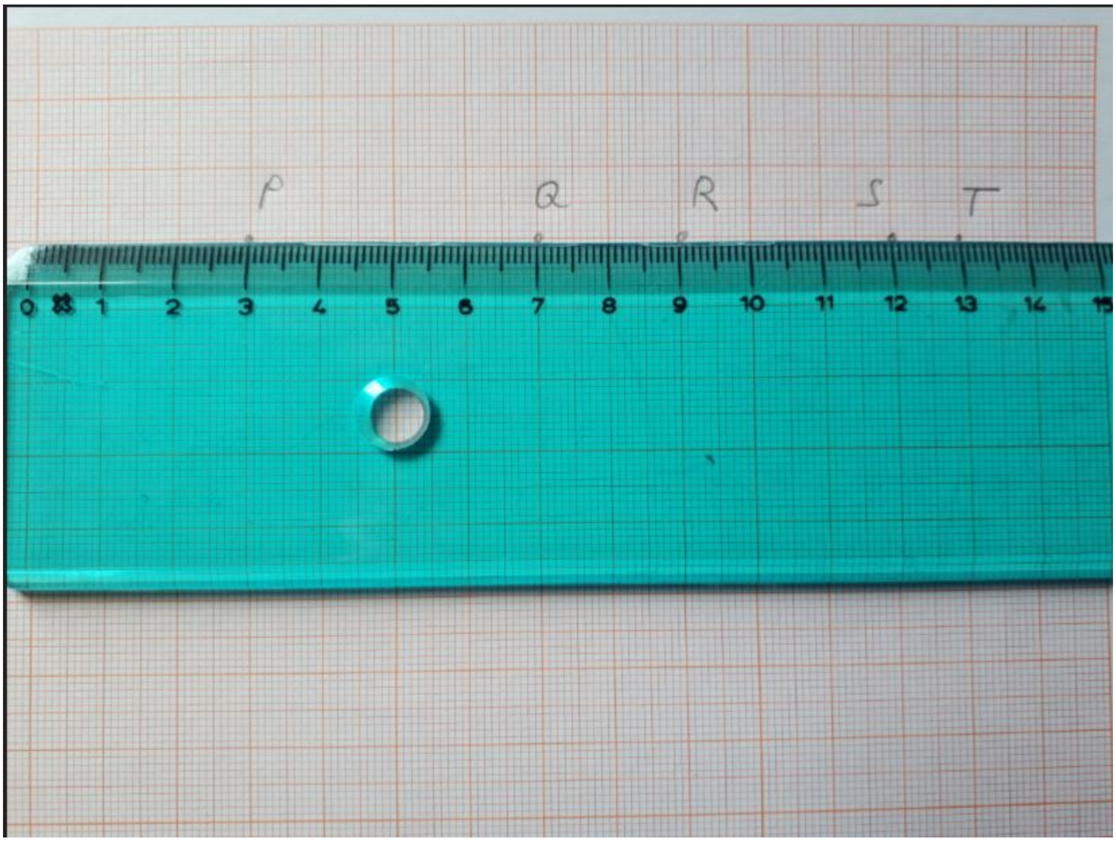


FIGURE 1.3: Points in line, the position is just the distance from the 0 of the rule.

At this point, if we ask the students to compute the distance between two points with the same abscissa or ordinate, they will surely note that the formula is reduced to the subtraction of the position.
It is important that the students understand the difference between the geometrical description and the algebraic one. The geometrical description is bounded to the ability to draw and every distance is measured directly, indeed the reference system is unnecessary and in the first exercise, we have not drawn it. Instead, the algebraic description requires only to know positions as real numbers, and every distance is computed from positions so the usefulness of the system of reference is undeniable even if it is not necessary. Furthermore, it can be said that there is a *function* that given two points returns the distance between them, in other words, the concept of metric should be introduced. This is particularly easy if the students have some basic training in programming, as it is slowly becoming customary thanks to many experimental activities on coding that are performed even in primary schools. The metrics may be considered as a function of a program, given the input (the points coordinates) it will perform various operations and then it will return the distance between the points.

$$(\ coordinates\ of\ P\ ;\ coordinates\ of\ Q\ ) \rightarrow distance\ between\ P\ and\ Q \tag{1.2}$$

It is important to note that we can think and talk about points in space without referring to any reference systems, but that they are really convenient. A more complete treatment of this topics, with the metrics fist presented as I did, can be found in [7].
A great advantage of the algebraic description is how well it leads to generalization. It is easy to generalize the formula for the distance to any number of dimensions

$$d^2(P;Q) = \sum_i (q_i - p_i)^2 \tag{1.3}$$

with obvious symbol meaning.

Another exercise to propose the students is to compute the distance of two points in different reference systems, as shown in 1.4

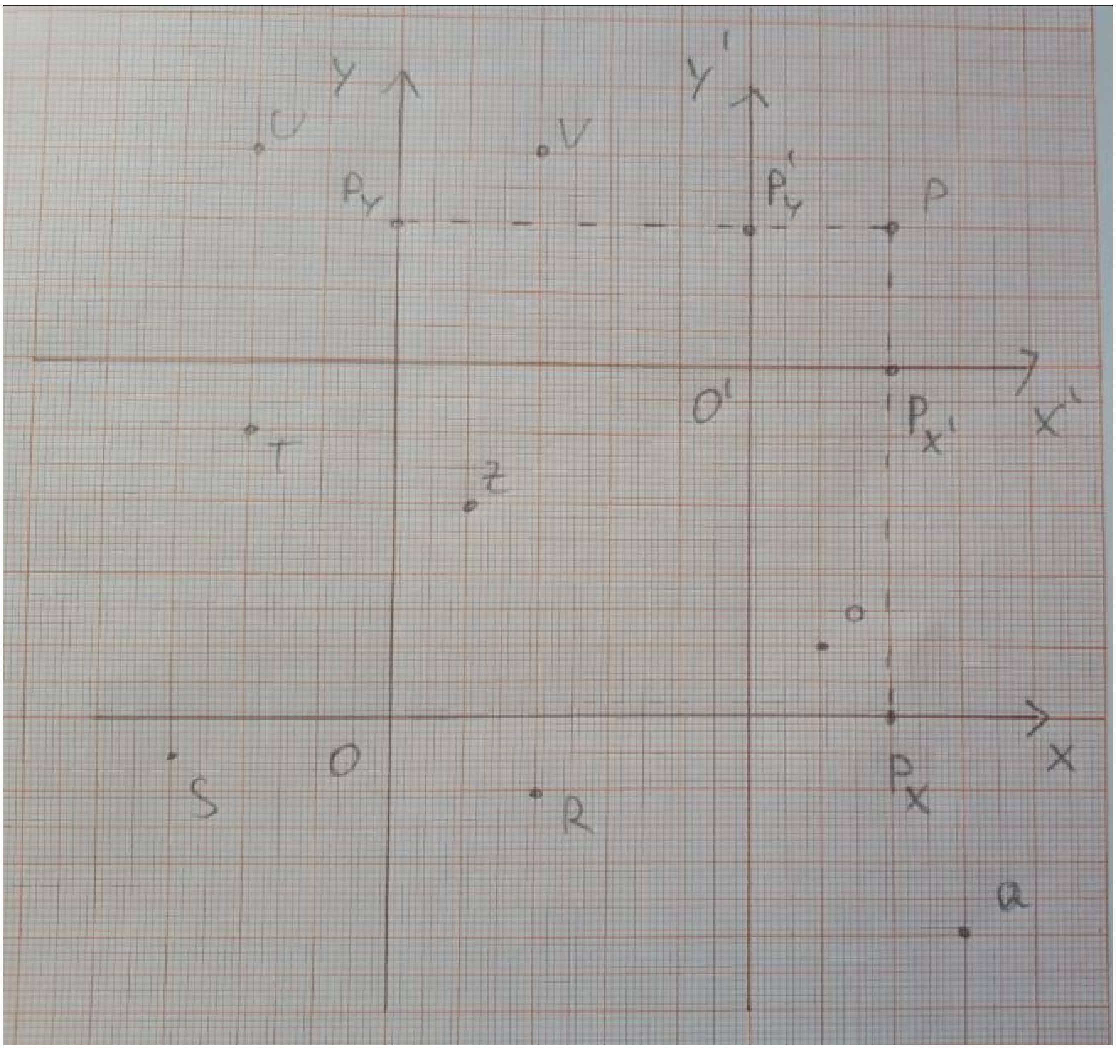


FIGURE 1.4: Two Cartesian coordinate systems where the students can exercise in finding the coordinates of points.

In the geometric description is intuitive to assume that the distance between points does not depend on the reference systems, but in the algebraic version this is not obvious. A few trials will persuade the students that this is the case, giving the opportunity to show how something can be *proved* with a calculation. The first thing to do is to understand how the coordinates of a point change when we pass from a reference system to another, as in the example before were the reference systems are shifted with respect to one another. In this case, ones just have to add or subtract a quantity to the coordinate of every point to change reference systems, in other words

$$x' = x + a$$
$$y' = y + b$$

and this leads to the distance in the second frame as

$$d(Q;P) = \sqrt{x_1'^2 + y_2'^2} = \sqrt{[x_1 + a - (x_2 + a)]^2 + [y_1 + b - (y_2 + b)]^2} = \sqrt{x_1^2 + y_2^2}. \tag{1.4}$$

Then the independence of the distance from the frame of reference in the algebraic descriptions follows from the fact that $c - c = 0$.
It is slightly more complicated to repeat the computation when the reference system is tilted, as shown in 1.5

If the students have no familiarity with the trigonometric functions, this is a good time to introduce them. Nowadays this topic is frequently proposed in the first year of high schools as an operation that given an angle returns a real number and as a way to project segments. In figure 1.6 the cosine is the number one has to multiply the $\overline{OP}$ length to obtain the length of $\overline{OP_x}$ while the sine is the number by which one has to multiply $\overline{OP}$ length to obtain the length of $\overline{OP_y}$.

Of course this number depends on α and some numerical example to show the value of sine and cosine at particular angles would help students understand them. Of course this is not a rigorous and satisfactory definition of the trigonometric functions, nevertheless to give first an incomplete but intuitive definitions and to refine it when the needs arise is a technique used even in university textbook like [8]. For a discussion on the importance of definitions in mathematical teaching, we recommend [9]. For a laboratorial experience on the Snell-Descartes law that can be used to introduce the sine and cosine we recommend the *Project LES* (Laboratori per l'Educazione alla Scienza) on http://www.les.unina.it/.

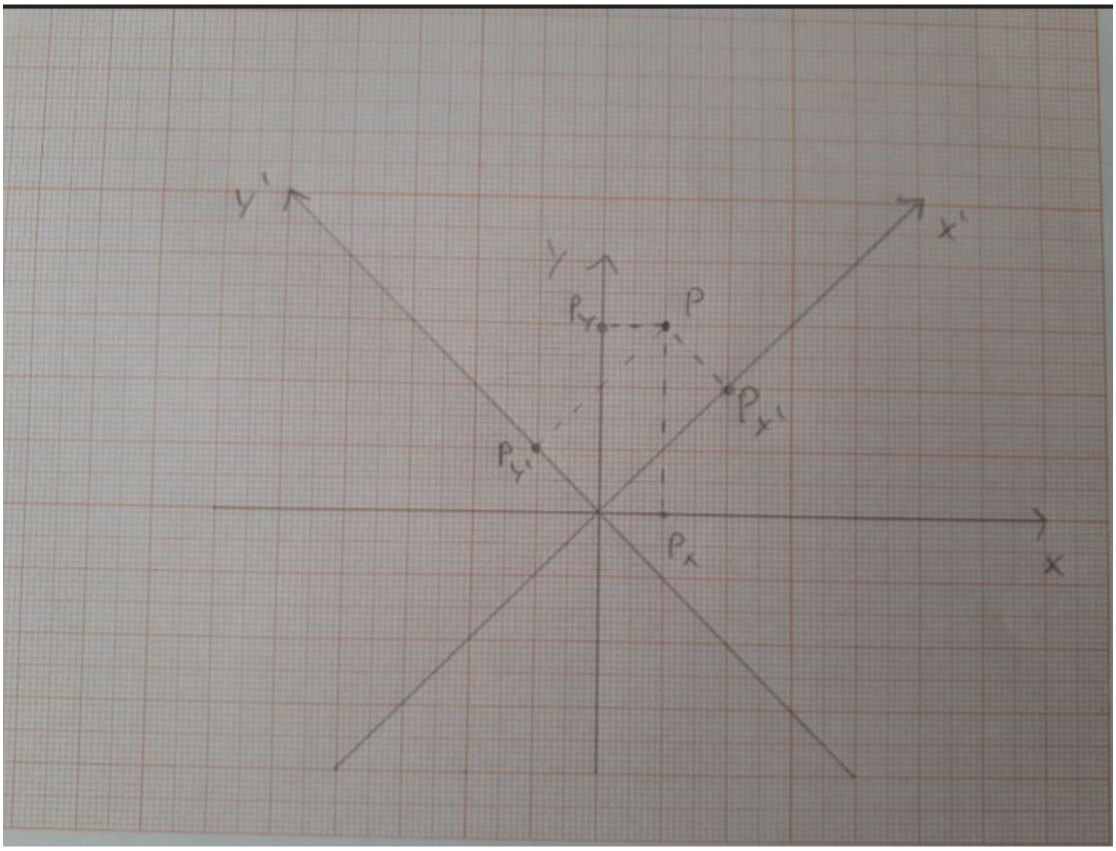


FIGURE 1.5: Two Cartesian coordinate systems tilted one with respect to the other of an angle α. Students can try to measure the coordinates of points in the two references.

Turning back to the reference systems, some work in the geometrical description allows to find the formulas

$$x' = x\cos\alpha + y\sin\alpha$$
$$y' = -x\sin\alpha + y\cos\alpha.$$

If for simplicity we take the distance from the origin of the system of reference,

$$d^2(O;P) = x'^2 + y'^2 = (x\cos\alpha + y\sin\alpha)^2 + (-x\sin\alpha + y\cos\alpha)^2 = x^2 + y^2 \tag{1.5}$$

Now the transformation for a roto-translation are intuitive,

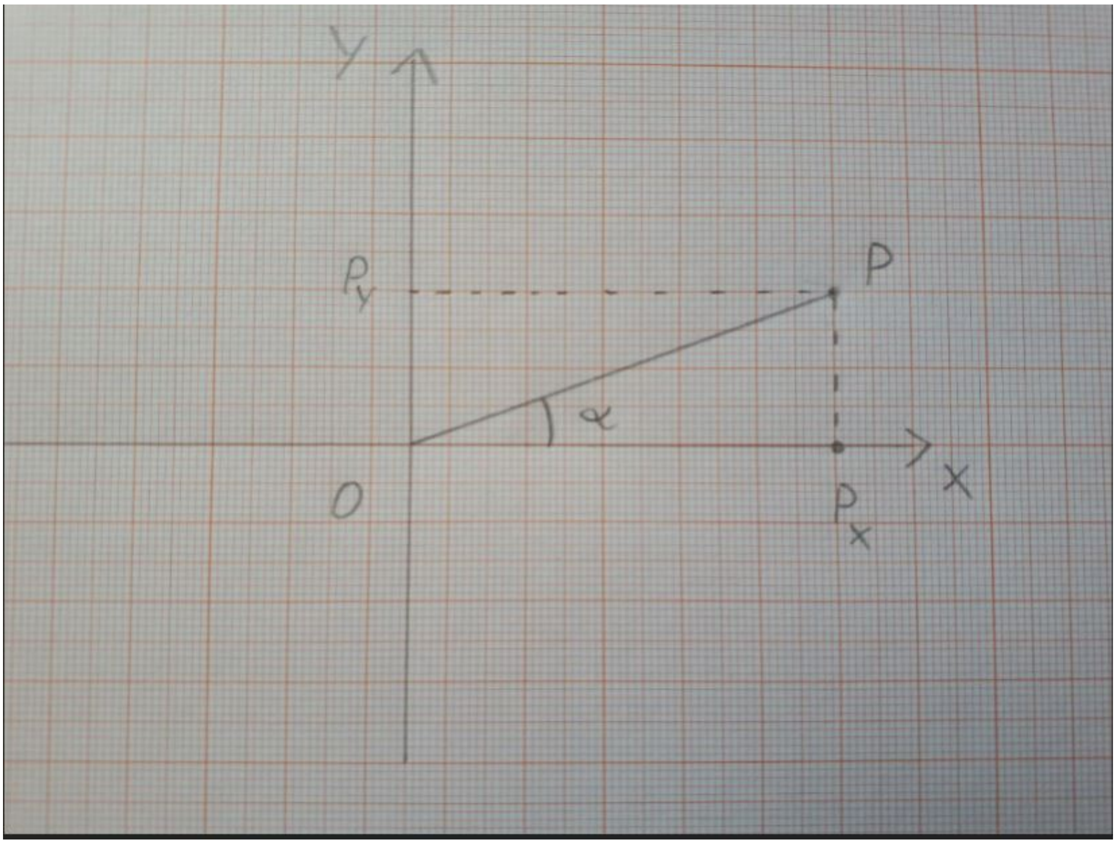


FIGURE 1.6: Orthogonal projections of a segment can be used to introduce the sine and cosine functions.

$$x' = a + x\cos\alpha + y\sin\alpha$$
$$y' = b - x\sin\alpha + y\cos\alpha.$$

In other words, in a change of reference systems one must combine *both* coordinate of the old reference to obtain *one* coordinate in the new reference, and the distance between points is *invariant* under this transformation. As with the distance, we can look at the roto-translations as mere operations, like objects of a program. Given the parameters for the translation a and b and the angle of rotation α, one performs the operations to tell the new coordinates. Inter alia, rotations can be expressed in the algebraic notations with matrices. For the moment being, one can introduce them just as tables on which is possible to multiply one another allowing to rewrite the transformation of a rotation as

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \tag{1.6}$$

The discussions on how matrix behave under change of frames or the definition of determinant may wait for the moment, but it could be useful to ask students to represent *translation* in this new *formalism*. An exercise like this could help students understand how to *work* with formalism and investigate its power and its weakness, in the specific that a matrix assoiciated to a transformation must have the origin fixed. In this way mathematics is not presented as results, but as a structure that can be built which has a great impact on the proficiency of students, like shows the works like [10] and [11]. We stress out the importance of this approach to the new concept since is often argued that it is not possible to teach modern physics without the mathematical preparatory concepts and that is not possible to teach mathematics if the students are not ready to grasp the definition and the rigorous thought needed. We think that the opposite is true, it is not possible to teach mathematics without an intuitive description while intuition and physics help to build a more rigorous thought-pattern until abstraction is recognized as the most efficient way by the students.

1.2 Plane geometry and optics

Doc: REACH!
Engineer : IS THIS A HOLD-UP?
Doc: IT IS A SCIENCE EXPERIMENT!

Back to the future

The pentalaser, shown in the figures below, is simply made of five lasers whose rays are parallel with *good* accuracy. For *good* we mean that even if the rays are projected on a screen several meters away, the light dots remain at the same distance with the precision of the millimetre. In this particular tool, made by *Kvanta* we can choose to light up only the mid laser, all five lasers, the central laser and two on the sides. The pentalaser can be combined with printed sheets and big lenses made with Poly(methyl methacrylate), also known as Plexiglas, to illustrate the functioning of optical systems. I have utilized this instrument with different audiences, from middle school to university. For students of the first year of the degree course in *Ottica e Optometria* provided by the Department of Physics *Ettore Pancini*, the activity was suggested by their professor *Antonio Sasso* during my work as a tutor. It has been a way to visualize in a crystal-clear way Snell Descartes law, fostering their interest in the subject. For middle-class students, it has been a way to understand how lenses and mirrors works.
For now and for our purpose, an angle will be the thing we measure with the goniometer. Then we can add a prism to the configuration, to measure the angle by which the ray is deflected as shown in 1.7.

The great power of this methods is that it visually connect everyday phenomena with their description in the frame of optical geometry. As in the already cited laboratorial experience of the L.E.S., it is possible to mount the prism on a rotating plate in order to measure the dependence of the diffraction angles from the incident angle, the Snell-Descartes law.
Another experiment that could be done collaborating with the biology teacher is shown in 1.8.

The light emitted by an object far away, is schematized as a bunch of parallel rays of light. The crystalline is schematized as a lens that converges the ray on the retina. This is the case of a perfect eye, but we can study what happens when the eye is too long or too short, as shown in the figures 1.9, 1.10, 1.11, 1.12.

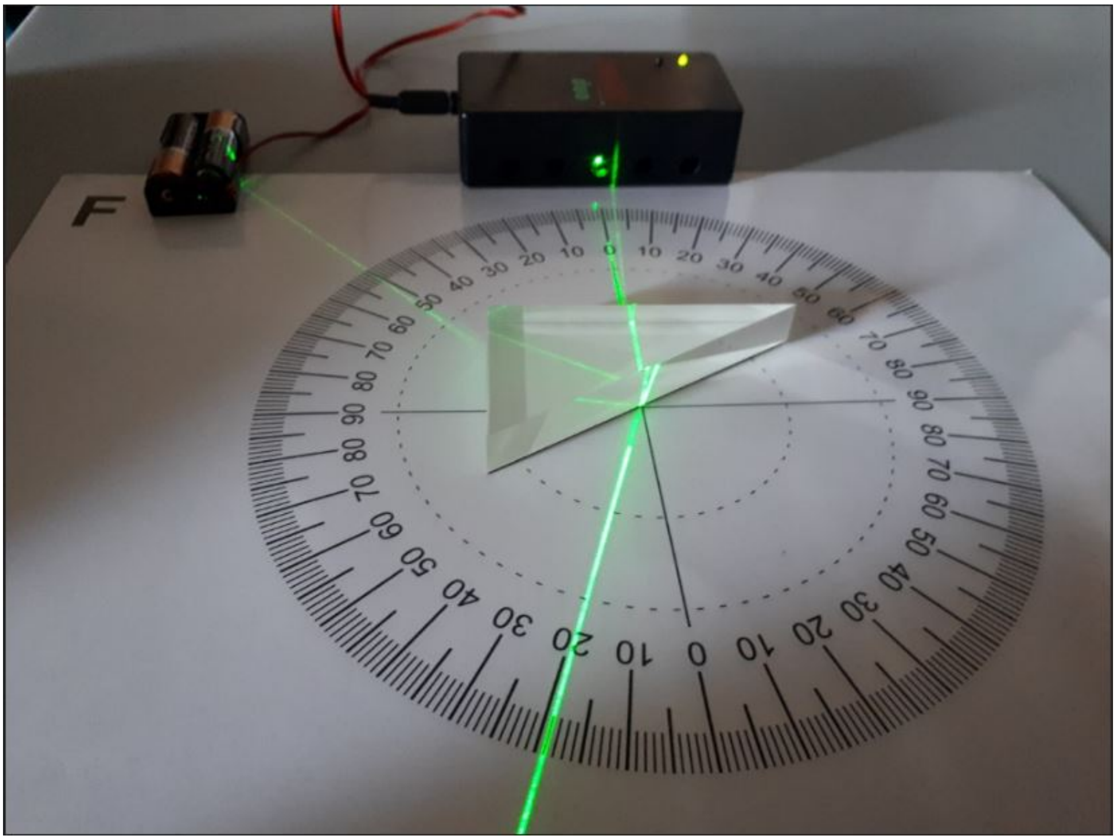


FIGURE 1.7: The prism refraction of light

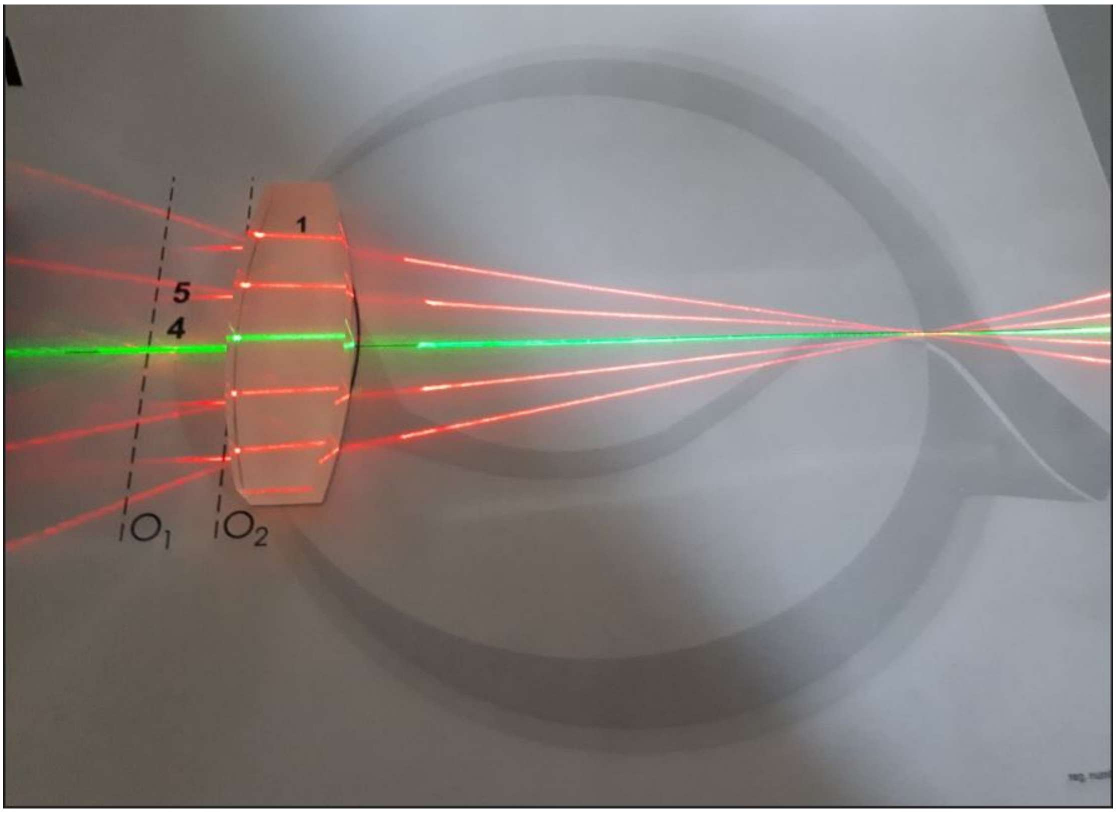


FIGURE 1.8: How the crystalline converges the ray on the retina

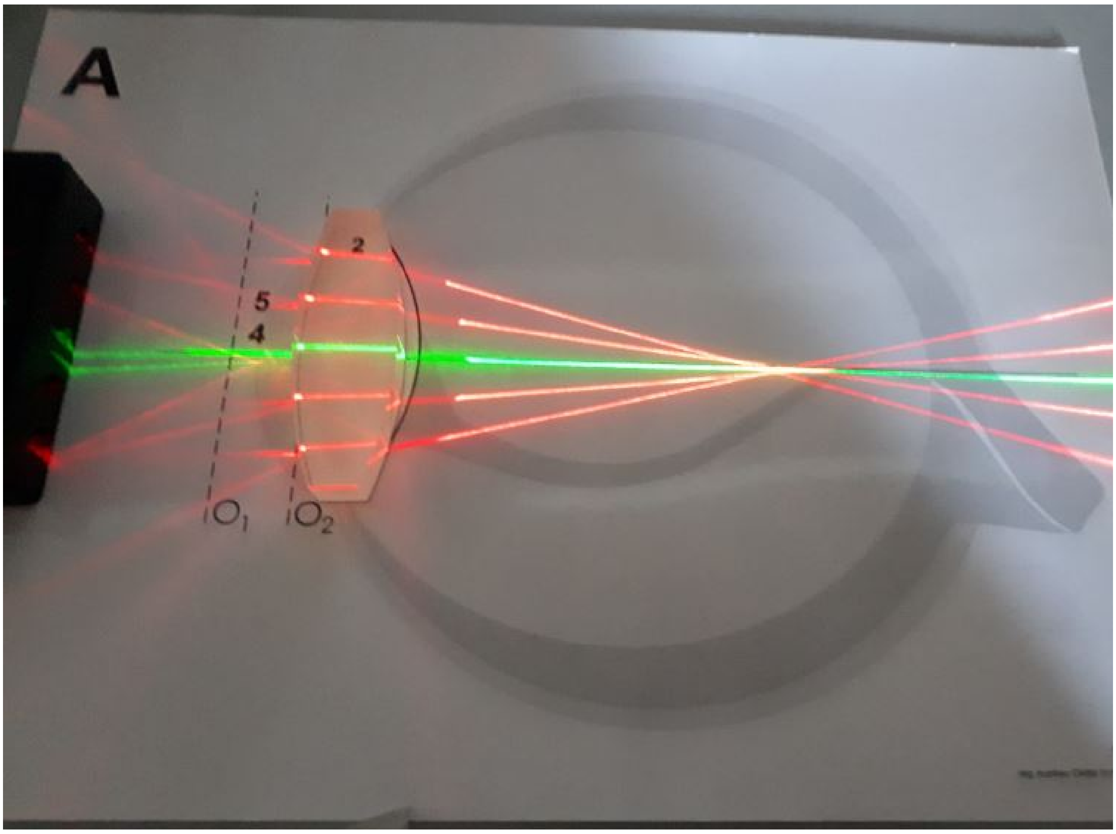


FIGURE 1.9: A miopic eye is too long and the rays of light converge in front of the retina.

If the eye is too long the rays converge in a point placed before the retina (in the experiment apparatus we have used a different lens as crystalline). To correct this defect, we add a divergent lens. If the eye is too short, the rays converge in a point placed after the retina. To correct this defect, we add a convergent lens.
Showing how the physical models describe actual problems and can be used to solve it is very powerful. It worked both with middle school students that with university students, fostering the comprehension of a symbolic language and its translation in actual phenomena.
The experiments can go even further, explaining the functioning of the telescope or even discussing the aberrations which are omitted here for brevity. Instead, a very interesting activity that can be done is the demonstration of the second law of Kepler as shown in [8]. This will emphasize the connection between astronomy and geometry. Moreover, also the other two Kepler's law can be derived trough geometric arguments toghther with Newton theory of gravitation, as can be see in [12].

Imagine that during a given time interval a planet moves from the point P_1 to P_2 as showed in 1.13. If during another time interval of the same duration it moves with the same velocity, it will arrive at P_3. The triangles $P_1\hat{S}P_2$ and $P_2\hat{S}P_3$ have equal areas since

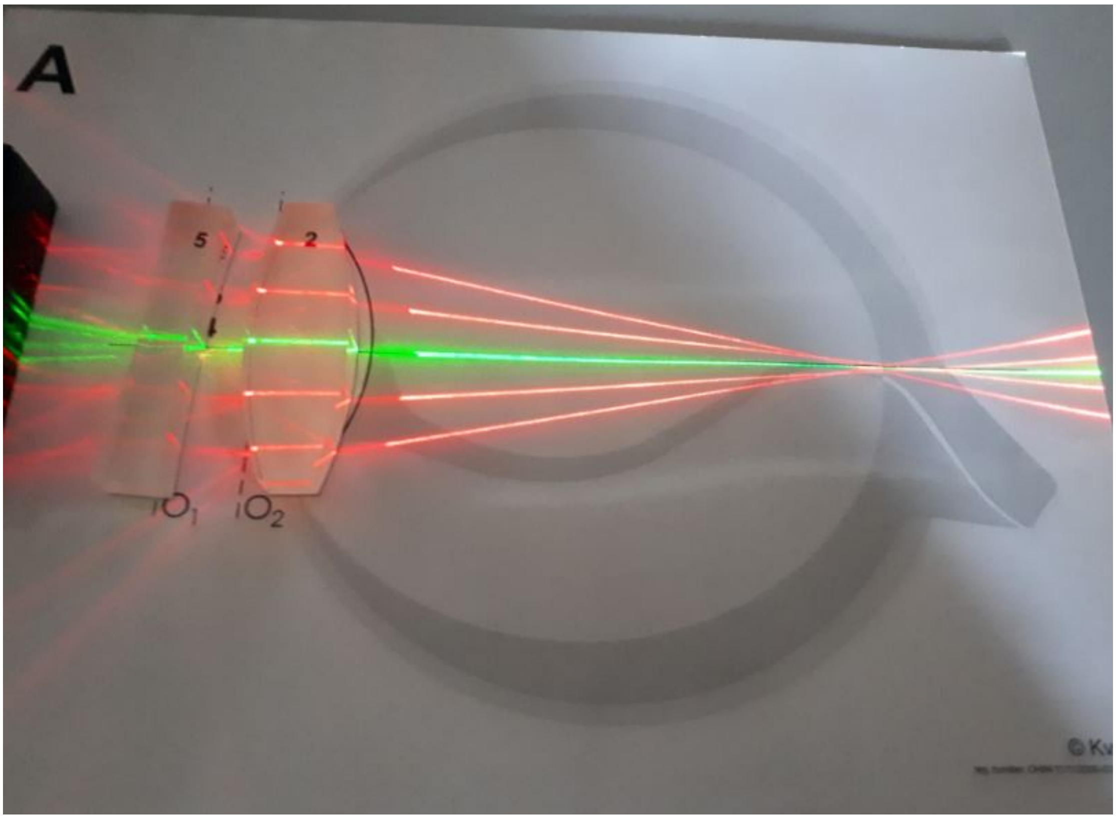


FIGURE 1.10: A divergent lens helps the crystalline to focus a bunch of rays of light on the retina.

they have the same base and the same height. This is the demonstration that if a planet moves with constant velocity and no other body attracts it, the segment that joins the planet with the point S sweeps equal areas in equal intervals of time. Now, imagine that the Sun at S attracts the planet. When the planet is in P_2 a force attracts it but instead of moving to the point Q, it will arrive in R as a consequence of the combination of the two motions. Thus the area swept in the second interval of time is $P_2\hat{S}R$ but, since the segment $\overline{RP_3}$ is parallel to the segment $\overline{SP_2}$, it has the same area of $S\hat{P_3}P_2$ and hence the area from the joining segment from the planet to the sun sweeps equals area in equal times, as we wanted to prove.

Of course, we should now make a limiting process in which the time intervals and, in tunr, the distances will be taken smaller and smaller and the actual planet trajectory will appear to be continuous. In a subsequent section we shall compute the trajectory of the planets.

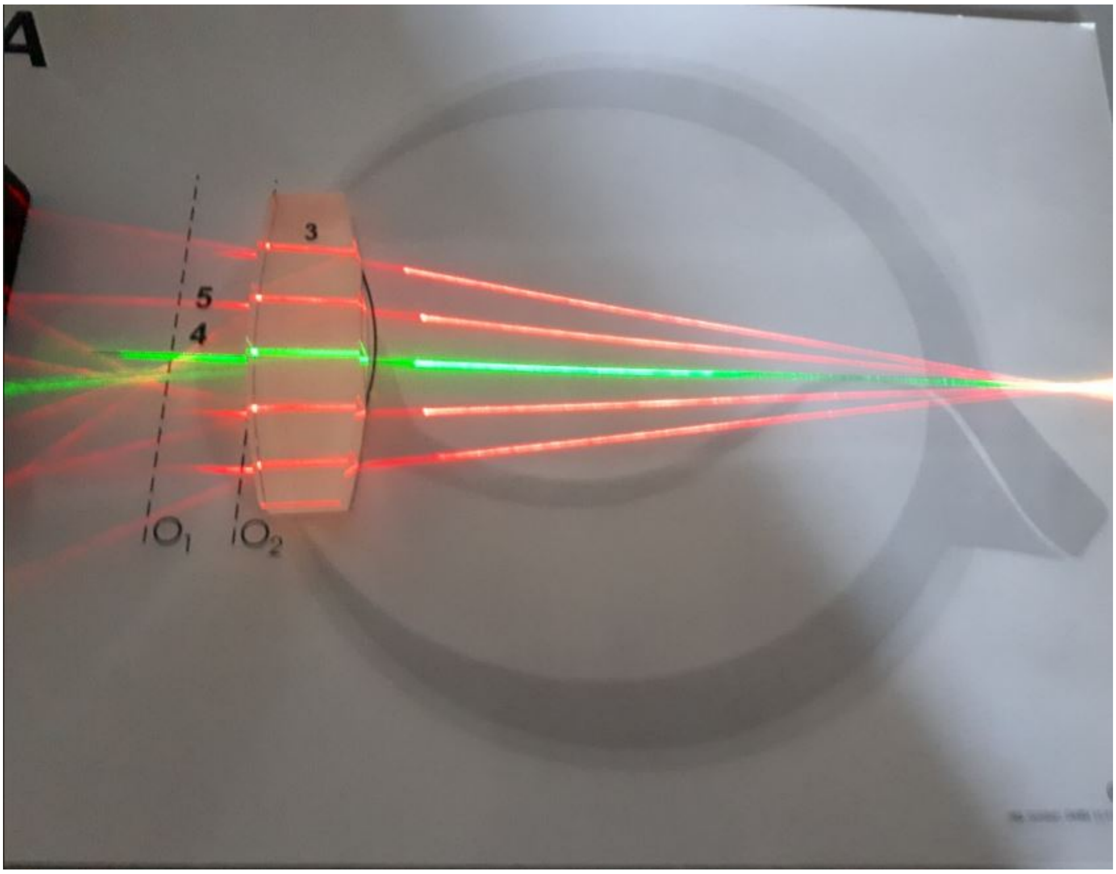


FIGURE 1.11: A farsighted eye is too short and the rays of light converge behind the retina.

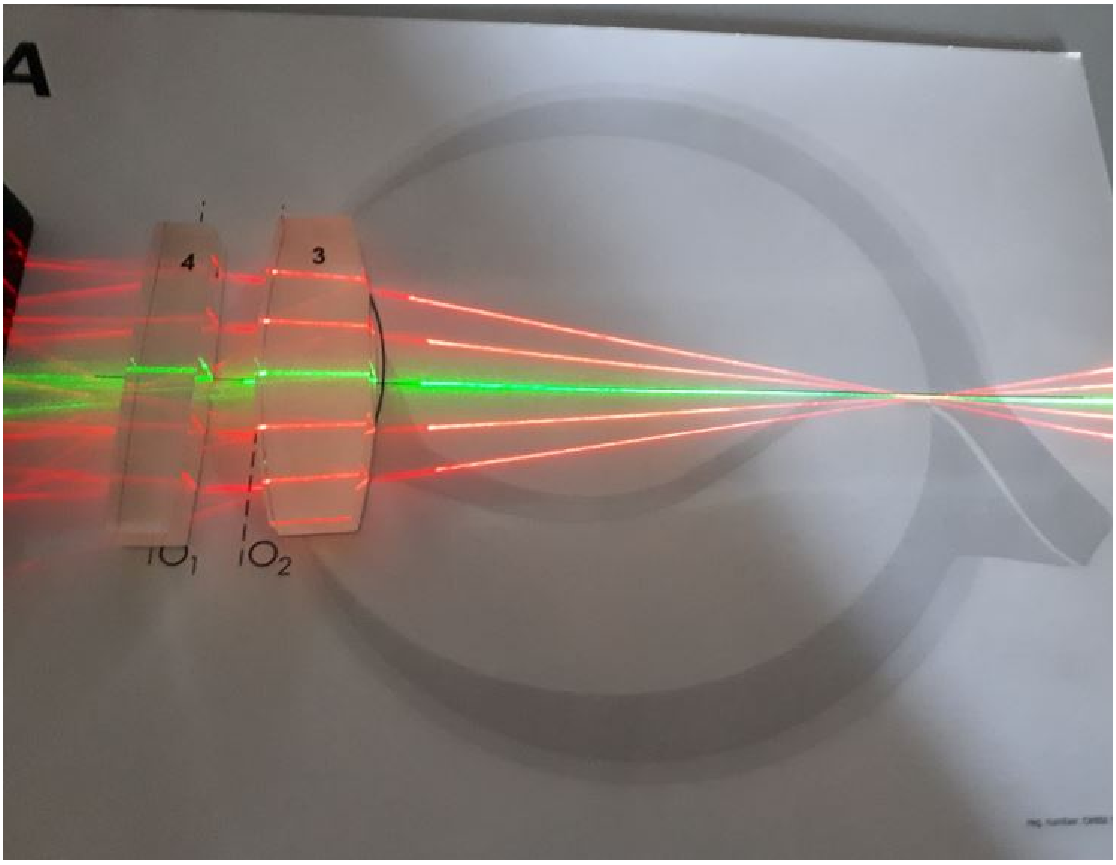


FIGURE 1.12: A convergent lens helps the crystalline to focus a bunch of rays of light on the retina.

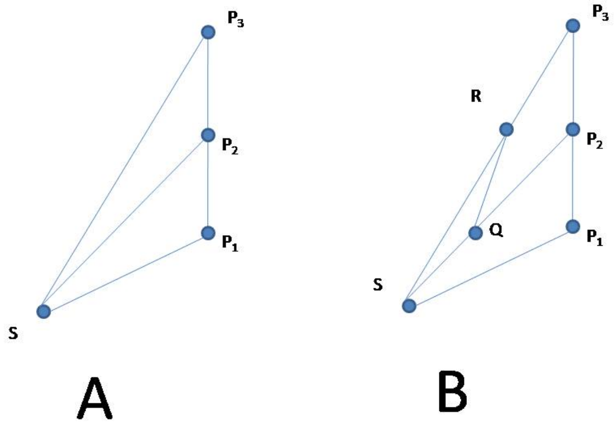


FIGURE 1.13: A schematisation of the planets trajectory. In fig. A the there is no gravitational force while in fig. B the sun is placed in S

1.3 How time was understood in different times

> the First Question, the oldest question in the universe, that must never be answered, hidden in plain sight.
>
> Dorium Maldovar to Doctor Who

In this section, we will look at how the concept of time has evolved to emphasize the link between society and science.
Science is often seen as cold and distant, students often ask *why do I have to learn it? when will I ever use it in real life?*. This happens because we do not need scientific knowledge to do anything. Someone else *who has that knowledge* has given us all the solutions. It suffices to think of one of the most common devices of our time, the smartphone. Behind it, a team of electrical engineers, data scientists, mathematicians, physicist and many others have worked, but the user-friendly interface makes it usable even to a four years old child. This is why it is so important to stress out the consequences of the scientific knowledge upon society and how society needs to drive scientific research. Indeed many textbooks like [13] and [14] start with a description of the most advanced techniques to measure physical quantities. As stated in [15] there is a very pragmatic reason if men started measuring times longer than days and lunar months, they needed to know *when* to sow and *when* to harvest. For example, the priests of ancient Egypt, probably the custodians of the more advanced scientific knowledge of their time, had the life or death task of looking out for the heliacal rising of the star Sirius. The heliacal rising occurs annually when the star becomes visible above the eastern horizon for a brief moment before sunrise, after a period of less than a year in which it had not been visible. In the same period of the year, a combination of favourable circumstances causes the Nile's floods. The Egyptians probably thought that stars influence actually modified earth phenomena, but this is a good example of how *correlation did not imply causation.* Anyway, if the floods where foretold they could have been used to fertilize the fields, if not they could have destroyed them. Furthermore, when society needs grew up and Egyptians needed to divide the day and night in hours to accomplish trading activities, instruments like the meridian and the nocturnal showed in 1.14 where invented.

Briefly, society cannot advance without science, but science takes is goals and methodologies from society, as is evinced in the work [16].
Since society needed accurate measurements of time, science provided it. Indeed, in order to measure time a complex definition of a reference system is needed. Astronomers took advantage of the great uniformity in Earth rotation. A point is chosen on the celestial

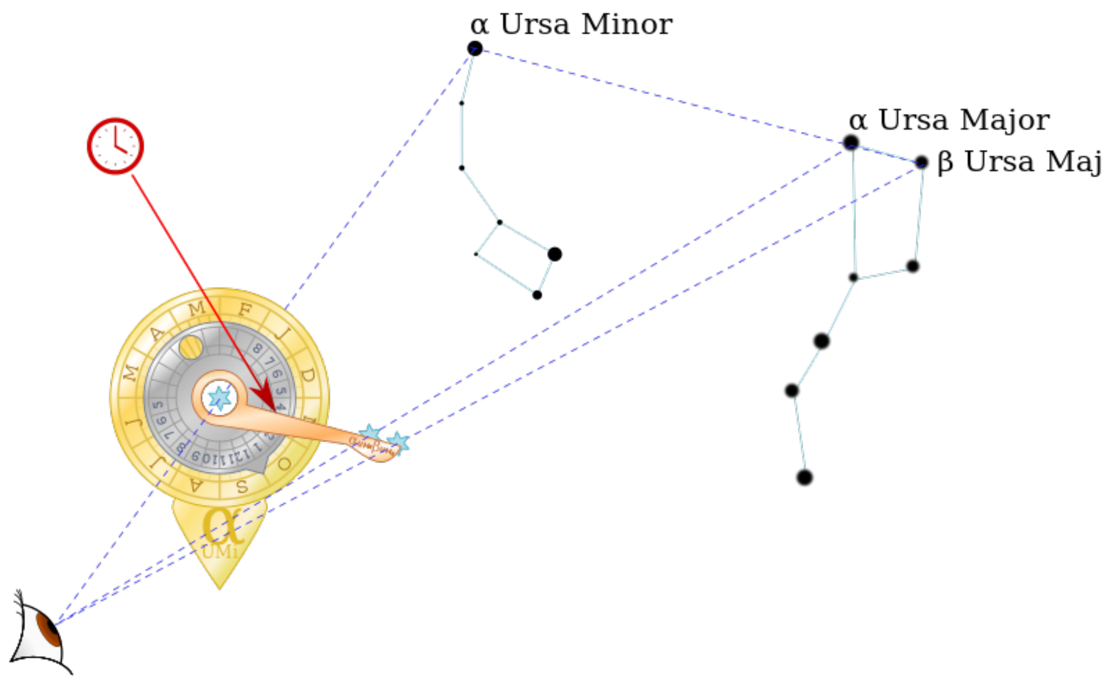


FIGURE 1.14: Functioning of a nocturnal - © nojhan/Claire Laudy — CC-BY-SA.

sphere, like the centre of the sun or the point on the ecliptic where the Sun crosses from the southern celestial hemisphere to the northern, which occurs at the (northern) vernal equinox, called vernal point. A measure of time was actually a measure of the degree by which one of this points had moved since in a day they perform a rotation of 360 degrees is easy to define the sidereal time and the solar time. For a more detailed discussion on astronomical systems of reference and time measurement, we suggest [17] or any astronomy textbook.

Anyway, this is why time and angles share so many common features like the term *seconds* which indicate both a time than a division of the grade.

Going on with the years, let us analyze the following problems taken from a medieval textbook with their solutions as translated in the work [18]. It will be evident that time was not thought of as a parameter and were not used while solving problems.

There is a field 150 feet long. At one end is a dog and at the other a hare. The dog chases when the hare runs. The dog travels 9 feet in a jump, while the hare travels 7 feet. How many feet will be travelled by the pursuing dog and the fleeing hare before the hare is seized? (Alcuin, 2005, p. 68)

The length of the field is 150 feet. Take half of 150, which is 75. The dog goes 9 feet in a jump. 75 times 9 is 675; this is the number of feet the pursuing dog runs before he seizes the hare in his grasping teeth. Because in a jump the hare goes 7 feet, multiply 75 by 7, obtaining 525. This is the number of feet the fleeing hare travels before it is caught.

(Alcuin, 2005, p. 68)
This problem can easily be solved with kinematic, but in the middle ages, they solved it without never explicitly considering time. Another example is this:
A fox is 40 paces ahead of a dog, and three paces of the latter are 5 paces of the former. I ask in how many paces the dog will reach the fox. (dell'Abacco as translated by Arrighi, 1964, p. 78)
The medieval solution to this problem starts considering that three-step of the dog are equals to five steps of the fox, $3D = 5F$. Then applying the proportion $5D = (8 + \frac{1}{3}F$ and the dog regains $3 + \frac{1}{3}$ fox paces at every step. Applying again a proportion, the number of steps the dog have to take are 60.
Of course, both problems and the last one, in particular, can be solved more easily introducing the time as a variable. But apparently, in the middle age, people did not think time as a variable. The jump or the steps are the measures of time since they happen *at the same time*.
This suggests that the more intuitive quantity is not time, but the movement. Indeed a *good* clock can be assumed to be something that appear to move with uniform speed. Following this line of reasoning, in the next section, we will study the proprieties of motions with the aid of the motion detector.
A good idea would be to repropose these problems with students in the classroom. One could tell students to make paces of given length, so that the idea that the steps are *simultaneus* and that they define the time can emerge spontaneously.
Going on with the years, times emerge as a *a-priori intuitions.* has described in [19], we distinguish two way of measuring time. One is, following the tradition of the astronomical definition, the search for systems that appears periodic, like pendulums. For example, we can compare two or more pendulum counting their oscillations. If the ratios of this number are constant, we recognize periodicity in the oscillations of the pendulums. Then, we can use pendulums as clocks, defining the oscillation of a particular pendulum as the unit of measure of time. Furthermore, this system is easy to reproduce and, for what we know, there are no variations in the oscillations frequencies, while the motion of the celestial bodies is subjected to secular variations. The other approach is to find systems that appear to vary uniformly, for example, a bottle full of a liquid with a hole in it on a weight scale. If the bottle is large enough so that for small times the water level does not vary too much, the water flows away uniformly. The variation in weight reported on the scale can be used as time. We cannot say that *the amount of water that flows away in a given interval of time is always the same* since we are trying to define time. The only way to test our approximation, that the water flows away uniformly, is to check our clock with others. In this picture, we assume that time exists and flows independently of our attempts to measure it.

In everyday life, we still use this naive understanding of time. Even when we use instruments like the GPS, where the computation on our position must be corrected with general relativity, we still think the time as something that passes uniformly and periodically. As we will see in the next chapters, this ideas must be rethinking if we want to understand the theory of relativity.
As for now, the more recent definition of *second* is:
The second is the duration of 9192631770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the caesium 133 atom."
13th CGPM (1967/68, Resolution 1; CR, 103)
"This definition refers to a caesium atom at rest at a temperature of 0 K."
(Added by CIPM in 1997)

1.4 Galilean transformation

> You know that in nine hundred years of time and space and I've never met anybody who wasn't important before.
>
> Doctor who

The motion detector is a sonar, like the one shown in 1.15, that sends an ultrasound wave in a cone with an approximate wideness of 30 degrees and receives the echo.

FIGURE 1.15: The motion detector, image taken from https://www.vernier.com/products/sensors/motion-detectors/go-mot/.

Knowing the velocity of the wave, the software *logger pro* can calculate the distance of the object who has reflected the wave to the sensor. This way a space-time graph like the ones showed in 1.16 may be drawn.

These tools have been tested for many years in didactic activities, examples may be found in the aforementioned *L.E.S.*. The power of this tools is the connection between the movement of the body and the graph of a function. Furthermore, with logger pro, it is possible to compute the derivative of the space-time graph, the velocity-time and the acceleration-time graph. This is a good way to introduce the operation of graphical derivation and integration, a task often asked in the Italian maturity exam.
The graph above was realized with a student of the *Liceo Scientifico Silvestri* of Portici during a laboratorial activity. The activity lasted two hours with about fifty students

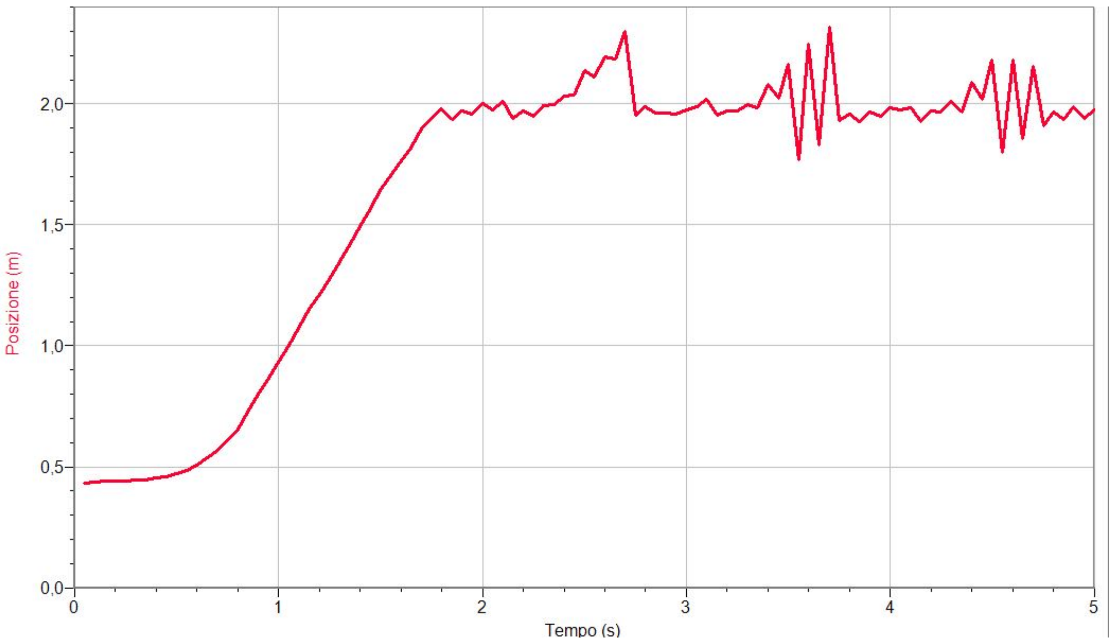


FIGURE 1.16: Space-time graph of a student.

from two classes. In the first part of the activity, the sonar functionalities have been illustrated recapitulating the rules of kinematics.

The students, who were in the third year of high school studies, where well prepared and the recapitulation has been smooth and quick, also they had already seen the motion detector in previous activities so they had no problem in reading the graph. They also recognized that the initial and final parts of the graph are not a straight line since the person starts moving and stops. As homework, we asked the students to *write down* a description of the various graph obtained with various people moving in a various way, focusing on rebuilding the movement of the person from the graph. In appendix A some of the student's work is collected and commented.

The second part of the experience was inspired by the already cited [18] and consists in taking a space-time graph *while the sensor is moving.*

When the sensor was *not* moving, after some trials the students made a straight line in the space-time diagram moving with quasi-constant velocity. To obtain a more (less) inclined line, they move faster (slower). Then we asked them if an object moves with a uniform speed, how will the space-time diagram look if *the sensor moves with a uniform velocity*? Some students predicted that the now graph should be a straight line but less inclined, since the *relative* velocity is *lower.* Indeed this is the case as shown in 1.17 and it is important to emphasize that *from the graph we cannot say if the sensor was moving.*

In the third and last part of the experience, we have formalized the concepts of Galilean transformation as a geometric transformation. A straight line in a given reference system

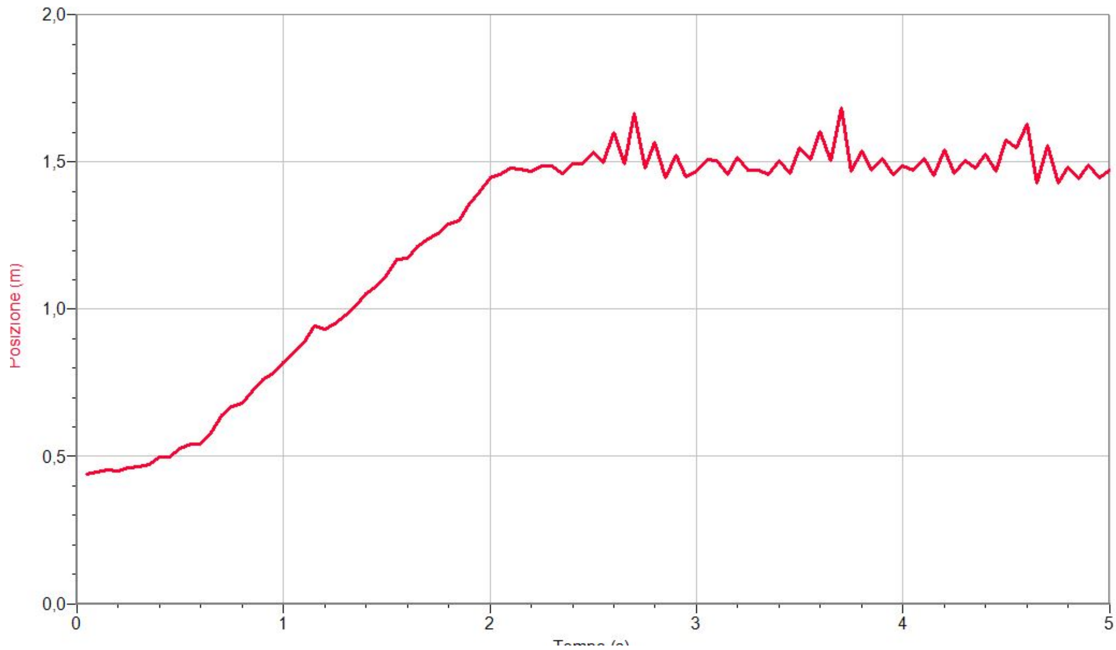


FIGURE 1.17: Space-time graph of a student with the sensor moved by another student.

where on the x-axis is represented *time* while on the y-axis is represented *distance* has an equation of the form $x(t) = x_0 + vt$ where x_0 is the initial position and v is the velocity of the object already introduced with the motion detector as the distance traveled in the given time unit. If we set our self in a different reference system $Ox'y'$ that *moves uniformly* with respect to Oxy, the line is still straight. It only changes the inclination of the line, in other words, the velocity and we call this transformation a *boost.* At this point I showed with a simple calculation *why* a straight line remains a straight line when the reference system is boosted. I started stating that we want to rewrite the equation of a straight line in a different reference system, (it is important to ascertain that students have already attained a structural comprehension of the equations, where the therm *structural* is to be intended as described in [5]). I proposed not to think the equation of the straight line as a way to compute x's and y's, but a mathematical object that given a reference frame Oxy can be expressed as $x(t) = x_0 + vt$ but in another reference system it will become $x'(t) = x'_0 + v't'$. To understand how the terms change I proposed the following exercises:

At the stadium, the audience clap their hands to support the home team. A defender, after recovering the ball, makes a long throw for the main striker and the balls advances of five meters every time the fans claps their hands. If the ball falls on the ground after eight claps, how many meters has the ball covered?

In this exercise, the students need to recognize the claps of the hands of the audience as the unit of measure for the time. Furthermore, they have to confine themselves in one dimension to avoid cumbersome computations.

At the stadium, the audience clap their hands to support the home team. A defender, after recovering the ball, makes a long throw for the main striker and the ball advances of fifteen meters. If the fans have clasped their hands eight times while the ball flew, what was the speed of the ball?

A midfielder makes a long throw forward. If the balls travel at a speed of thirteen meters per second and stay in the air for five seconds, at what distance from the midfielder the ball will fall?

With this two exercises, one can ascertain that the students have understood the concept of velocity.

Near the midfielder of the previous exercise there is a wing half who starts running to support the offensive scheme when the long throw is kicked. After how many seconds did he sees that the balls fall on the ground?

With this exercise, we want to emphasize that the time in which things happens does not change for people that move one with respect to the other. Both soccer players see that the ball falls in five seconds.

If the wing half of the previous exercise runs at a speed of six meters per second he will not get to the point on which the ball falls, how far from him will the ball fall?

Finally, this exercise shows that the position of points does change from a player to another. If we call x the point in which the midfielder sees the balls fall down on the ground, for the wing half the point in which the ball falls will be $x' = x - Vt$, where V is the velocity of the half wing.
We have then obtained the Galilean transformation

$$x' = x - Vt$$
$$t' = t$$

It is easy to generalize this transformation for the other spatial dimensions, it suffices to remember that the x direction is arbitrary. It is more interesting to show what happens to the equations $x' = x'_0 + v't'$ when change applies the Galilean transformation. In doing so one must remember that the initial position x'_0 turns into $x_0 - Vt_0$, where t_0 is the initial time that we can put to zero so that $x'_0 = x_0$ and the new velocity of an object is given by the velocity in the old reference system v minus the velocity of the reference frame V. We then obtain:

$$x' = x'_0 + v't'$$
$$(x - Vt) = (x_0 - Vt_0) + (v - V)t$$
$$x = x_0 + vt - Vt + Vt$$
$$x = x_0 + vt.$$

We have achieved a great deal that should be welcomed with much rejoicing. We have, by experience, found a symmetry that we think nature posses and we know how to write it in a mathematical form. In the next chapter, we will use this symmetry to build up mechanics, a theory used to build building, cars, aeroplanes, ships and many other things.

1.5 Non relativistic mechanics

> If all the parts of the universe are interchained in a certain measure, any one phenomenon will not be the effect of a single cause, but the resultant of causes infinitely numerous; it is, one often says, the consequence of the state of the universe the moment before.
>
> Henri Poincaré - The Value of Science

In the previous section we have seen how the space-time diagram of a object changes if we change the reference system with a Galilean transformation. The lines, not only the straight ones, are just rotated. We have not analyzed what happens if the new reference system is *accelerating* and we will not discuss this case now. So far the experience has told us that in a different reference systems moving with a velocity V the time-space diagram is the same, just rotated. In other words the numerical values of the quantities are changed, but the form of the equation (polynomial of degree n, exponential ecc..) are the same. This means that if we want to foretell the motion of an object with *mass* m, our prediction must respect this symmetry. The best way to do that is to use *vectors*. At school level vectors are introduced as *arrows* with *magnitude*, *direction* and *point of application*. But they can also be treated as an n-tuple of numbers *which transform in a specific way in a change of coordinate system*. For a more detailed treatment, we suggest [14], where it showed that in a change of reference system the coordinates of a vector change in specific ways so that the proprieties of the *object* as a whole are preserved. By the way, this is also a good reason to distinguish physical vector from arrays in coding, and by analogy table of numbers, matrix and tensors.
This suggests to use vector to express how object moves, so that our model will be invariant for Galilean transformation. With this in mind, lets start discussing the movement of an object of mass m. It is common sense that the greater the mass, the more difficult is to move the object and to sustain its movement. Usually at this point the three laws of dynamics are introduced, but we shall follow another route proposed by the *The Karlsruhe Physics Course*. The following line of reasoning has also been proposed in the ending comments of the activity in the Liceo Silvestri with good results, as teachers said it has connected various topics of physics which are often seen disconnected due to the asphyxiating rhythm of lessons.
We introduce the *momentum* as the product of the mass and the velocity of an object $\vec{p} = m\vec{v}$. Instead of the velocity we could have thought of position or acceleration, but

in this way this quantity would not preserve in time. If it was the product of the mass for the acceleration to be conserved, objects should always be accelerating. It was the product of the mass for the position, objects should stop, but unless a force like friction is applied on the objects they preserve their motion. Only the product of the mass for the velocity is preserved. Since the mass is an intrinsic propriety of the body, this is equivalent to the law of inertia. Moreover, if the momentum changes *something* must be changing it, and we give this something the name of *force* and write

$$\vec{F} = \frac{d}{dt}\vec{p} \tag{1.7}$$

or $\vec{F} = \frac{\Delta\vec{p}}{\Delta t}$ if students did not know the operation of derivation. This is the second law of dynamics, and it is also very clear that the variation in momentum is equal to the impulse. Lastly, the third law of dynamics is also a simple consequence of the principle of conservation of the momentum. If a body *gives* momentum to another body, its own momentum must decrease, getting negative if necessary.

The power of this equations is that it allows us to predict the actual evolution of a motion, and there is no need to actually solve the differential equation at all. As shown in [14], given the initial conditions one can compute the position and velocity after a short time interval of Δt. This action can be repeated to obtain the orbit of the planet. Of course, the gravitation force

$$|\vec{F}| = -G\frac{m_1 m_2}{r^2} \tag{1.8}$$

must be known. Furthermore, this force has to be projected on the axes, and as shown in [14] this can be done easily if one of the masses is in the centre of the reference system:

$$F_x = -G\frac{m_1 m_2}{r^3}x$$
$$F_y = -G\frac{m_1 m_2}{r^3}y$$

where x and y are just the coordinates of the planets. Always on Feynmann textbook, the generalization to the object in any points is found as:

$$F_x = -G\frac{m_1 m_2}{r^3}(x_1 - x_2)$$
$$F_y = -G\frac{m_1 m_2}{r^3}(y_1 - y_2).$$

In the Feynman lectures on physics, this algorithm is carried on by hand reporting the values on a table. Nowadays, it can be fastened using a *excel* sheet or a similar program, which is really intuitive and powerful. Nevertheless, the culture of coding is spreading from elementary schools. In Italy, a particularly successful project is described on the website $platform.europeanmoocs.eu/course\_coding\_in\_your\_classroom\_now$. To children, coding is presented as a game, often a video game, but cycle and conditional structure are learned at a very early age. Furthermore, the modern programming languages like Python or Mathematica are intuitive, to know how to code is a powerful tool. From my personal experience, to know how to code since the high school helped me in the study of physics just as much as calculus and linear algebra, the one who finds the numerical approach difficult often have never seen a program in their previous studies. To learn a programming language is like learning a mathematical notation, it is like a lens that puts a problem in a completely different light. Since coding is becoming more popular and since I believe that is an irreplaceable instrument, the following numerical studies are not carried out with an excel data sheet, but with the Python programming language and the Jupyter notebook. More information about Python and the Jupyter notebook can be found in appendix B, here it suffices to say that the language is so high level to resemble pseudo-code and the notebook allows to interact with it intuitively and effectively. Furthermore, we suggest [20] to the reader who wants others, more detailed examples of numerical methods in didactic. Differently from us, the cited work goes into the detail of the stability of the algorithms. We have used a variant of the Velocity-Verlet algorithm, which is just a little more complex than the Euler methods but it is more stable.

The following code plots the trajectory of Mercury around the Sun and the energy trend. The Sun is at the centre of the reference system and the action of the planet on him is neglected since the force is the same but the mass of the sun is seven orders of magnitude bigger. The algorithm starts with the initial position and velocity of the planets as taken from the N.A.S.A. fact sheet. To compute the position and velocity at the subsequent step, differently on how this was implemented by Feynmann textbook, the acceleration is computed from the gravitational force and the motion is supposed uniformly accelerated with that acceleration, which is a good approximation for small intervals of time. Furthermore, if the program is executed in the textitJupiter notebook a widget is launched and the trajectory may be tracked step by step.

```
from math import sqrt
from matplotlib.pyplot import *
from ipywidgets import interact
%matplotlib inline

def mercury(N = 101): #Define function mercury the number of iteration as a parameter


    t = [0] #The array of times, Dt is a hundredth of a year on mercury expressed in seconds
    Dt = 76005.216

    G = 6.67408*10**-11  #The gravitational constant expressed in m^3 kg^-1 s^-2
    M = 1988500*10**24  #The mass of the sun M and mercury m as reported by NASA factsheet
    m = 0.33011*10**24

#To set initial condition we referred to the NASA fact sheet,
#https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html
#The simulation starts with the planet in his aphelion.

    x = [69.82*10**9]            #Initial position
    y = [0]

    vx = [0]                     #Initial velocity
    vy = [38.86*10**3]

    r = [sqrt(x[0]**2 + y[0]**2)]
    v = [sqrt(vx[0]**2+vy[0]**2)]

    K = [0.5 * m * v[0]**2] #Initial kinetic, potential and total energy
    U = [-G * M * m /r[0]]
    E = [K[0]+U[0]]

    ax = [-G*M*x[0]/r[0]**3]
    ay = [-G*M*y[0]/r[0]**3]


    #In the for cycle, a new element is appended to the arrays

    for i in range (1, N):  #Start the cycle, the true hart of the algorithm

        x.append(x[i-1] + vx[i-1] * Dt + 0.5 * ax[i-1] * Dt**2)
        y.append(y[i-1] + vy[i-1] * Dt + 0.5 * ay[i-1] * Dt**2)

        r.append(sqrt(x[i]**2+y[i]**2))

        ax.append(-G*M*x[i]/r[i]**3)
        ay.append(-G*M*y[i]/r[i]**3)

        vx.append(vx[i-1] + 0.5 * (ax[i-1] + ax[i]) * Dt)
        vy.append(vy[i-1] + 0.5 * (ay[i-1] + ay[i]) * Dt)
```

```
        v.append(sqrt(vx[i]**2+vy[i]**2))

        K.append(0.5 * m * v[i]**2)
        U.append(-G*M*m/r[i])
        E.append(U[i] + K[i])

        t.append(t[i-1] + Dt)    #Advance time

    #Plot the results

    figure()    #Plots the trajectory of the planets
    plot(x, y)
    title("Trajectory, both axes have meters as units")
    grid()


    figure() #Plots energy as a function of time
    plot(t, E)
    plot(t, K)
    plot(t, U)
    title("Kinetic, potential and total energy")
    xlabel("Time[s]")
    ylabel("Energy [J]")
    grid()

    show()


interact (mercury, N = (1,100000,1))
 #The program is interactive so the trajectory can be followed step by step
```

The output of the program is showed in 1.18 and 1.19

This shows that the trajectory of Mercury is an ellipse with the Sun in one of the focuses. This means that Kepler laws follow from Newtonian laws, which are more general, more useful and more powerful. It is interesting to note that certain quantity can be introduced whose sum is constant during all the motion, kinetic and potential energy. This quantity gives much information about the system that would be hard to retrieve in a different way. To show the power of the concept of energy as a tool to model natural phenomena, let's examine for a moment a simple system. On an inclined plane, two bottles are rolling down, one is empty and the other one is half full of sands. From cinematic proprieties, all object should fall with the same velocity, but the sand inside the second bottle slows it down. This is easily explained if we think that the gravitational potential energy is divided into two channels: rolling down the bottle and shaking the sand.
Turning back to numerical methods, they are so powerful that allow us to study systems that cannot be solved analytically. In the following code, the trajectories of both Mercury and Jupyter are tracked taking into account their mutual interaction.

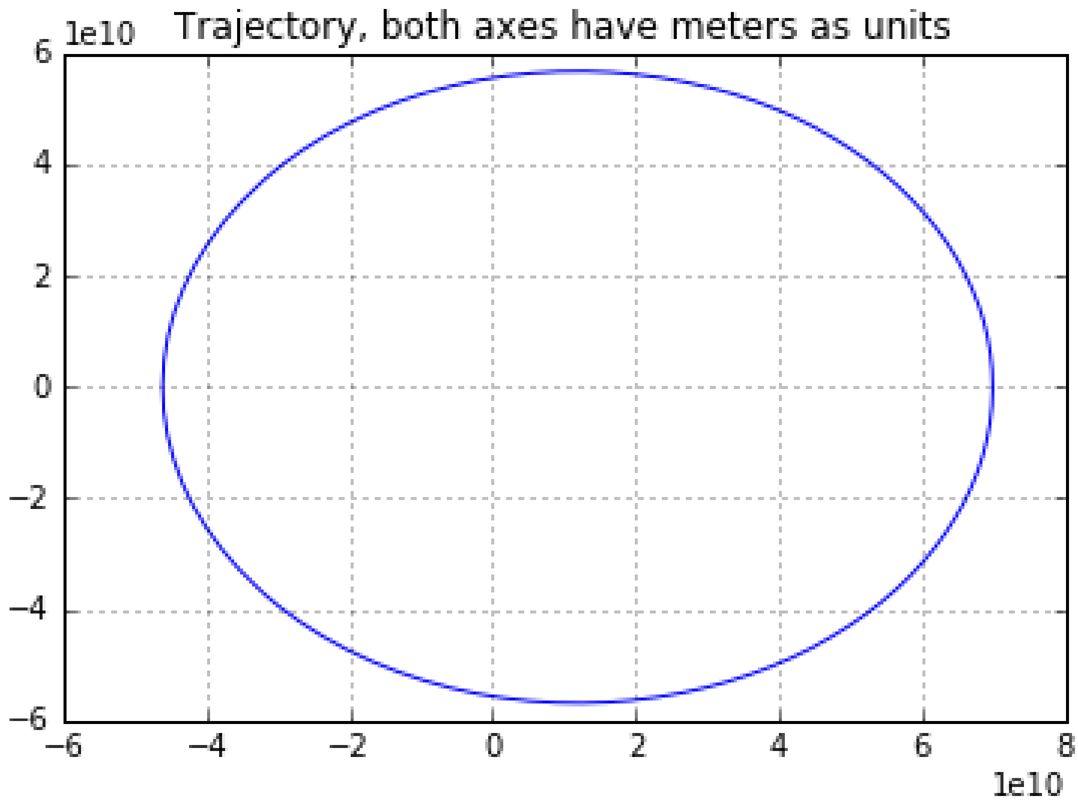


FIGURE 1.18: Mercury trajectory.

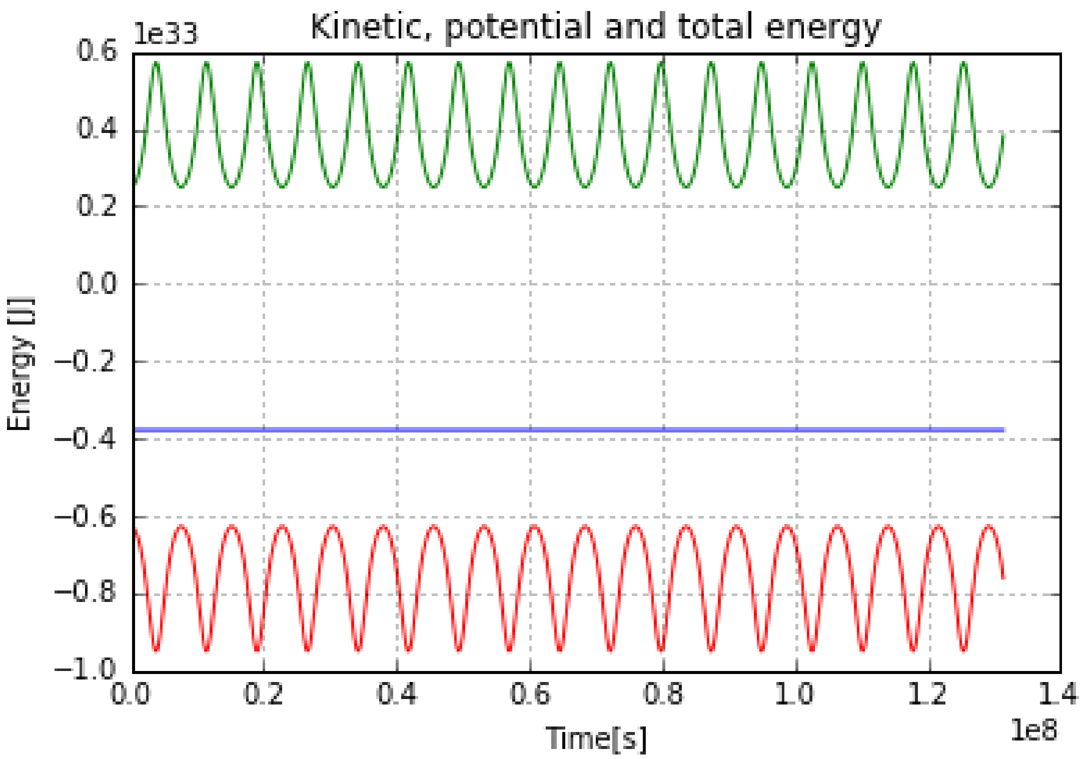


FIGURE 1.19: In red the potential energy, in green the kinetic energy and in blue the total energy. This graph is not obtained with $N = 101$, any *period* of kinetic and potential energy is a complete revolution of Mercury around the Sun.

```
from math import sqrt
from numpy import size
from matplotlib.pyplot import *
from ipywidgets import interact
%matplotlib inline

def mercury(N = 3000):


    t = [0]
    Dt = 76005.216

    G = 6.67408*10**-11
    M = 1988500*10**24
    m = 0.33011*10**24
    J = 1898.19*10**24

#To set initial condition we referred to the NASA fact sheet,
#https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html
#The simulation starts with both planets aligned in the aphelion.
#That is a simplification since the orbits of the planets are inclined
#one with respect to the other.

    xm = [69.82*10**9]
    ym = [0]

    vxm = [0]
    vym = [38.86*10**3]

    xJ = [816.62*10**9]
    yJ = [0]

    vxJ = [0]
    vyJ = [12.44*10**3]

    rm = [sqrt(xm[0]**2 + ym[0]**2)]

    rJm = [sqrt((xJ[0]-xm[0])**2+(yJ[0]-ym[0])**2)]

    rJ = [sqrt(xJ[0]**2 + yJ[0]**2)]

    axm = [-G*M*xm[0]/rm[0]**3 - G*J*(xm[0]-xJ[0])/(rJm[0]**3)]
    aym = [-G*M*ym[0]/rm[0]**3 - G*J*(ym[0]-yJ[0])/(rJm[0]**3)]

    axJ = [-G*M*xJ[0]/rJ[0]**3 - G*m*(xJ[0]-xm[0])/(rJm[0]**3)]
    ayJ = [-G*M*yJ[0]/rJ[0]**3 - G*m*(yJ[0]-ym[0])/(rJm[0]**3)]

    for i in range (1, N):

        xm.append(xm[i-1] + vxm[i-1] * Dt + 0.5 * axm[i-1] * Dt**2)
        ym.append(ym[i-1] + vym[i-1] * Dt + 0.5 * aym[i-1] * Dt**2)

        rm.append(sqrt(xm[i]**2+ym[i]**2))

        xJ.append(xJ[i-1] + vxJ[i-1] * Dt + 0.5 * axJ[i-1] * Dt**2)
```

```
        yJ.append(yJ[i-1] + vyJ[i-1] * Dt + 0.5 * ayJ[i-1] * Dt**2)

        rJ.append(sqrt(xJ[i]**2+yJ[i]**2))

        rJm.append(sqrt((xJ[i]-xm[i])**2+(yJ[i]-ym[i])**2))

        axm.append(-G*M*xm[i]/rm[i]**3 - G*J*(xm[0]-xJ[0])/(rJm[0]**3))
        aym.append(-G*M*ym[i]/rm[i]**3 - G*J*(ym[0]-yJ[0])/(rJm[0]**3))

        axJ.append(-G*M*xJ[i]/rJ[i]**3 - G*m*(xm[0]-xJ[0])/(rJm[0]**3))
        ayJ.append(-G*M*yJ[i]/rJ[i]**3 - G*m*(ym[0]-yJ[0])/(rJm[0]**3))

        vxm.append(vxm[i-1] + 0.5 * (axm[i-1] + axm[i]) * Dt)
        vym.append(vym[i-1] + 0.5 * (aym[i-1] + aym[i]) * Dt)

        vxJ.append(vxJ[i-1] + 0.5 * (axJ[i-1] + axJ[i]) * Dt)
        vyJ.append(vyJ[i-1] + 0.5 * (ayJ[i-1] + ayJ[i]) * Dt)

        t.append(t[i-1] + Dt)


        #Plot the relusts

    figure()
    plot(xm, ym)
    plot(xJ, yJ)
    title("Motion")
    grid()
    show()

interact (mercury, N = (1,100000,1))
```

The output of the code is shown in figure 1.20

This can be generalized to any number of interacting objects, it suffices to generalize the formula for the decomposition of the gravitational force as

$$F_x = -\sum_i^N G\frac{m_i m_j (x_i - x_j)}{r^3}$$
$$F_y = -\sum_i^N G\frac{m_i m_j (y_i - y_j)}{r^3}.$$

We can even include the third spatial dimension or the effects of the planets on the Sun that causes the star to move, a movement that in others stars allow us to detect planets orbiting them!
Other formidable tools to study the interaction between many bodies can be found on the site https://phet.colorado.edu/en/simulations which is full of applets that can be

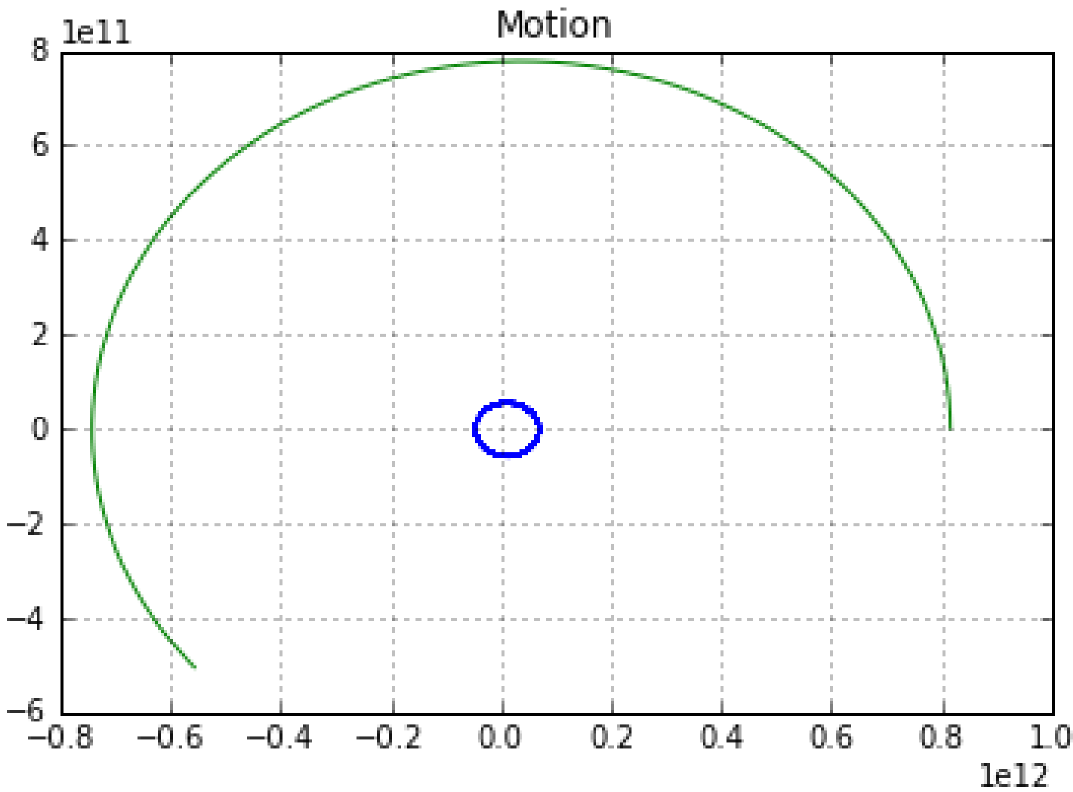


FIGURE 1.20: In blue Mercury trajectory, in green Jupyter trajectory

inspected in classroom or at home by students.

We end this chapter with a provocative question: It is the Earth that turns around the Sun or vice versa?

The question is ill-posed. The description of the Earth rotating around the Sun is simple, everything follows from the conservation of momentum and the gravitational force. A description of the Sun rotating the Earth is mathematically feasible but to obtain a model that fits with experimental facts, Coriolis force, millennial motions etc... many *ad hoc* terms should be added to the model. A better question would be, if we put a camera outside the solar system, would we see the Earth rotating around the Sun?

Once the mechanics are treated as we suggest here, it is easy to generalize it to the relativistic case, as we will show in the next chapter.

Chapter 2

Special theory of relativity

> You're just not thinking fourth dimensionally!
>
> Doctor Emmett Brown - Back to the future

In this chapter, we will generalize the Newtonian mechanics to relativistic mechanics. The first section is a semi-quantitative derivation of the wave equation for the electromagnetic fields from the Maxwell's equation in integral form. The second section is a recapitulation of the mechanical waves and a comparison with the electromagnetic ones. In the third section, we propose a laboratorial activity with the Michelson interferometer, followed by a description of the Michelson and Morley experiment followed by an introduction to the Lorentz transformation as a generalization of the Galilean transformation. In the last part of this section, non-relativistic mechanics is presented in an analogous way as the Newtonian mechanics was introduced in the last chapter.

2.1 Derivation of wave equation from Maxwell's equations

> The Scientist must set in order. Science is built up with facts, as a house is with stones. But a collection of facts is no more a science than a heap of stones is a house.
>
> Henri Poincaré - Science and Hypothesis

The Maxwell's equations are known in high school in the integral form. The *Gauss* law:

$$\Phi_S(\vec{E}) = \frac{Q}{\epsilon_0} \tag{2.1}$$

On the right side of the equation, we have the flux of the electric field $\Phi_S(\vec{E})$ over a closed surface S. To give an idea of what a flux is, we can use the hydrodynamical analogous to the flux of water. If a given quantity of water is moving with a speed of 1 metre per second, and it escapes from a pomp hole whose surface is $2cm^2$ (*this is not a closed surface*), the flux will be of $0.0002m^3s^-1$ or 0.2 litre per second. In other words, we can fill a glass in one second. If we cover part of the hole or insert an inclined plate in front of the hole, water will get out faster since the quantity of water taken from the source is the same, but it has to escape from a smaller hole. Anyway, if the inclined plate is imaginary or if we consider just half of the hole without actually closing the other half, the speed of the water does not change (assuming the fluid is incompressible), but the value of S does. This is what is usually done when applying Gauss law, we *imagine* the most convenient *closed* surface in order to solve a given problem.

Going back to the equation, on the right side we have the source of the electric field, the charge Q divided by the dielectric constant in vacuum ϵ_0. The charge takes the information on how much electric field is *generated*, like how much water is taken from the source but with the difference that is not the quantity that outsprings in a unit time, is just the quantity of electric field in the space, generated by the charge Q. We can justify the dielectric constant with two arguments, it is needed to assure consistency in the dimensions of the two sides of the equations and it remembers us that if in actual material, and not in the vacuum, the phenomenon is different and the equations will take this into account.

Gauss law is intuitive and reasonable, there is an amount of electric field in the space and the flux on a closed surface, like a sphere, depends only on the magnitude of the surface. Again, if we consider a *well behaved* fluid escaping a hole and we imagine concentric

spheres of different radius surrounding it, the flux of water for the different surface is the same. The quantity of water that goes out any sphere, in a given amount of time, is the same but as the surface of the spheres through which water passes gets bigger, the speed of the water diminishes. In an analogous way, as the sphere around a charge gets bigger, the intensity of the electric field diminishes. Furthermore, the flux and the field intensity must be in a proportional relation. Since the surface of the sphere gets bigger in proportion with the square of the distance from the charge, the intensity of the flux must decrease in the same way. We have assumed that the reader is a teacher with a basic or advanced knowledge in physics, so the reader knows that the electric field is defined as the force that acted on a unit charge, hence the Coulomb force must satisfy a relation with the square of the distance at the denominator. This is another way to emphasize how all pieces of a scientific theory fits together, science is true even if no one believes in it, but people (and researcher are people) must be convinced of the theory they are working on.
The second of Maxwell's equations states that the flux of the magnetic field on a closed surface is always zero, this can be explained assuming that magnetic charges (north and south pole) are always in pairs and the field line is closed, the equation is

$$\Phi_S(\vec{B}) = 0. \tag{2.2}$$

The third equations, also known as *Faraday-Lenz* law, connects the variation in time of the flux with the work needed to move a unit charge on a closed circuit Γ

$$\Gamma(\vec{E}) = -\frac{\Delta\Phi_S(\vec{B})}{\Delta t}. \tag{2.3}$$

In this case, the flux is to be considered on the surface that has as boundaries the circulation path and it is not closed.
The left-hand side of this equations is also known as *circulation* of the electric field. In the case of a flux that varies in a spire, the circulation is equal to the electromotive force generated in the spire and can be measured. This is the first of Maxwell's equation in which $\vec{E}$ and $\vec{B}$ are connected. Indeed, the first two equations give information on the fields but can be obtained from the other equations [14].
The last (but not least) of the Maxwell's equations, also known as the *Ampère circuital* law, connects the circulation of the magnetic field with the sum of all the electric currents I trough the path of circulation and the variation in time of the electric flux on the surface whose boundaries are the circulation path, in formula

$$\Gamma(\vec{B}) = \mu_0 \Big(I_{tot} + \epsilon_0 \frac{\Delta\Phi_S(\vec{E})}{\Delta t} \Big). \tag{2.4}$$

The term on which we will focus is the *displacement current* $\epsilon_0\mu_0\frac{\Delta\Phi_S(\vec{E})}{\Delta t}$. This was adjunct by Maxwell to adapt the equations in cases, like a capacitor, where there is no conduction current but the field is not zero. With this two equations, that connect the electric and the magnetic fields, it is possible to explain how many everyday objects works like electric engine or electromagnets, but we will focus on a different topic.
With this two equations, we can see that there is the chance to create a self-sustaining propagating electromagnetic field. The derivation that we proposed can is a sum up of the ones that can be found in [13] or [14], and also the pictures are taken from there. Consider an infinite sheet of charge moving forward, as shown in figure 2.1.

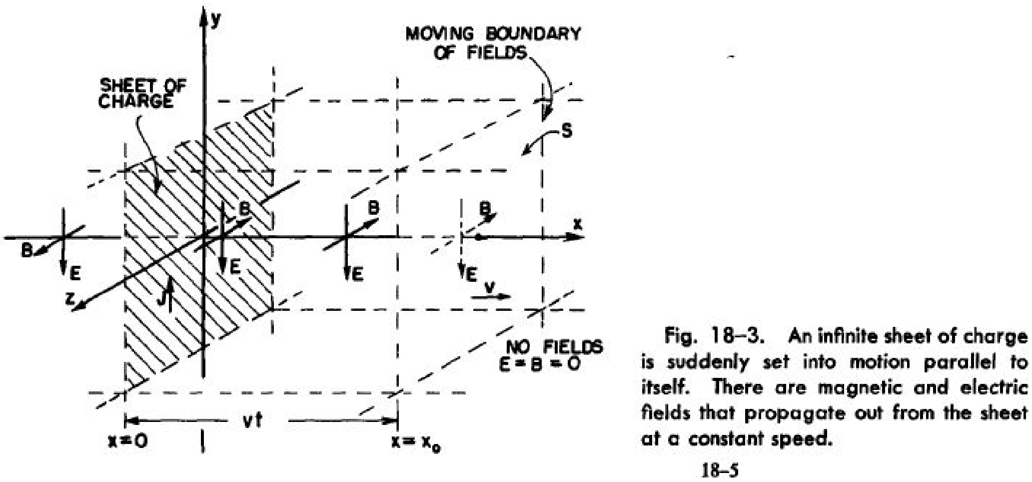


Figure 2.1: Image taken from "The Feynman Lecture on physics", aslo avaible on *www.feynmanlectures.caltech.edu/II_18.html*

As the sheets moves, a magnetic and an electric field are generated. We will apply the last two Maxwell's equations to this fields. To do so, we need to consider two section of the previous picture as shown in 2.2.

On the figure 18-5, we can apply the Faraday-Lenz law, the circulation is just EL, where L is the side of the rectangle which we chose a path for the circulation while the change in the magnetic flux is vBL where v is the speed of the perturbation. Given Faraday-Lenz law, $E = vB$. We have omitted the symbol of vectors because we are only interested in the magnitude of the vectors since the direction is given in the figures.
On figure 18-6 we can apply Ampère-Maxwell law, then in an analogous way the circulation is just BL, there are no electric currents but the change in the flux of the electric

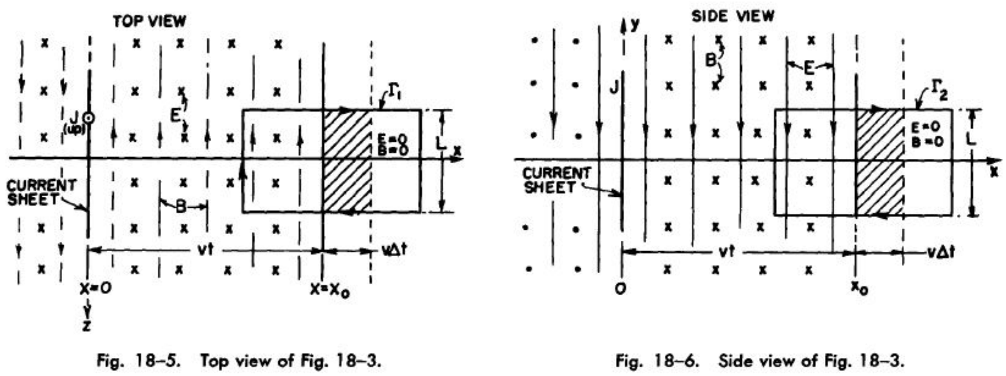


FIGURE 2.2: Image taken from "The Feynman Lecture on physics", aslo avaible on *www.feynmanlectures.caltech.edu/II_18.html*

field is $\epsilon_0\mu_0 vBL$. This leads us to $E = \epsilon_0\mu_0 vB$.

The two equations obtained are compatible with one another only if $v = \frac{1}{\sqrt{\epsilon_0\mu_0}}$, which is the speed of light c. This suggests looking at the electric and magnetic fields not as two different objects, but as two manifestations of the same object, the electromagnetic field. Moreover, this suggests that light is an electromagnetic wave.

To end this section, we want emphasize that the Maxwell equation are not invariant under Galilean transformation. This has not been treated with the students for a shortage in time, but Maxwell equations predicts that the speed of the electromagnetic signal is the same in any reference frame, as is shown in [14].

2.2 Comparing mechanical waves with electromagnetic waves and Michelson interferometer.

> People assume that time is a strict progression of cause to effect, but actually, from a nonlinear, non-subjective viewpoint, it's more like a big ball of wibbly-wobbly, timey-wimey... stuff.
>
> Doctor who

To test our idea in presenting electromagnetism and relativity, I have gone in the *Liceo Scientifico Carlo Urbani* to present two seminaries for two hours to three jointed classrooms of the last year. The students were already familiar with almost all of the concept I explained but, as stated by both students and teachers, to study again the same topics but in a different light, connecting arguments that were previously discussed in a different month or even different years has been a great stimulus.
In the first activity, I have presented a recapitulation of mechanical waves and a comparison with electromagnetic fields. The first object I have used is a slinky (see 2.6), with the help of this toy mechanical waves can be actually *seen*. Both longitudinal and transverse waves can be discussed and some laboratorial activity can be found on the L.E.S. My goal was to show that a wave is usually described in two ways, as

$$x(t) = A\cos\omega t + \phi \tag{2.5}$$

or

$$y(x) = A\cos kx + \phi. \tag{2.6}$$

What many students had not grasp was the meaning of ϕ, and some numerical example was helpful in clearing up the meaning of this term.
So in a mechanical wave *something* oscillates, and the direction of the oscillation is called *polarization*. For the wave on the slinky, we have found three possible directions of polarization, two for the transverse waves and one for the longitudinal. Of course, the various polarizations can be combined, but at this point with the help of a laser, two Polaroids and a slit, the wave nature of light has been shown (see 2.7). The proposed experiments are all well know and well described in the literature, and the students had already seen

them in videos proposed during the mandatory lessons.

FIGURE 2.3: Myself at the Liceo Scientifico Carlo Urbani explaining the sinusoidal wave.

At this point, to trigger their speculations, I asked: "if the electromagnetic radiation, as we have seen, is a wave, what is oscillating?".
After a brainstorm, following the line of thought of the previous chapter, we arrived at the conclusion that are the electric and magnetic fields that oscillate. With this, the first lessons has ended.
After a month, I returned to the school and meet the same student, taking from where we left. I bought the microwave apparatus showed in 2.9, where On the left there is an antenna that radiates in the frequency of microwaves, on the left the receiving antenna with a graduated scale to read the intensity of the current generated by the incoming radiation. In the middle, a support to add various material like polarizations plates, slit and double slits.
I have briefly proposed again the polarization and the double slit experiment, so to recap

FIGURE 2.4: Myself at the Liceo Scientifico Carlo Urbani explaining Lorentz transformation as a geometrical operation.

FIGURE 2.5: Myself at the Liceo Scientifico Carlo Urbani showing a dinamo to illustrate the magnetic induction.

FIGURE 2.6: Myself and a student at the Liceo Scientifico Carlo Urbani illustrating mechanical wave with the help of a slinky.

the previous lesson. After that it was really easy to explain the basic features of the Michelson and Morley experiment, that was already discussed in mandatory lessons.
The experiment is widely known in the literature, for our purpose it suffices to remember that the interferometer is composed of two orthogonal arms with mirrors at the end. A light beam is split with a semi-transparent plate at 45 degrees with respect to the light beam. The beams are then reflected back and recombine on a screen. Varying the optical path of the two beams, we expect a change in the interference figure on the screen. A typical Michelson interferometer is shown in figure 2.10 and 2.11. This one uses microwaves instead of light beams, and the fringes are observed measuring the maxima and minima on the scale of the receiving antenna.

Moreover, the Michelson interferometer can be build also with the microwave optics desk, as shown in figure 2.12 and 2.13.

FIGURE 2.7: Myself and some students at the Liceo Scientifico Carlo Urbani showing how two Polaroids can be used to stop the light.

FIGURE 2.8: Myself and some students at the Liceo Scientifico Carlo Urbani showing the diffraction of light from a single slit.

FIGURE 2.9: Myself at the Liceo Scientifico Carlo Urbani showing the functioning of the microwave apparatus. On the left is the radiating antenna, on the left the receiving antenna with a graduated scale to read the intensity of the current generated by the incoming radiation. In the middle, a support to add various material like polarizations plates, slit and double slits.

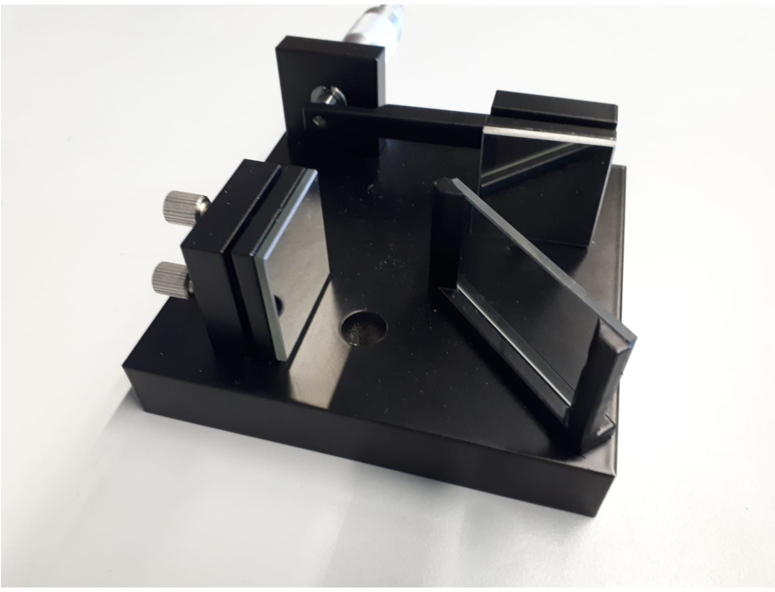


FIGURE 2.10: The beam splitter. Two micrometre screws allow a perfect alignment of the mirrors.

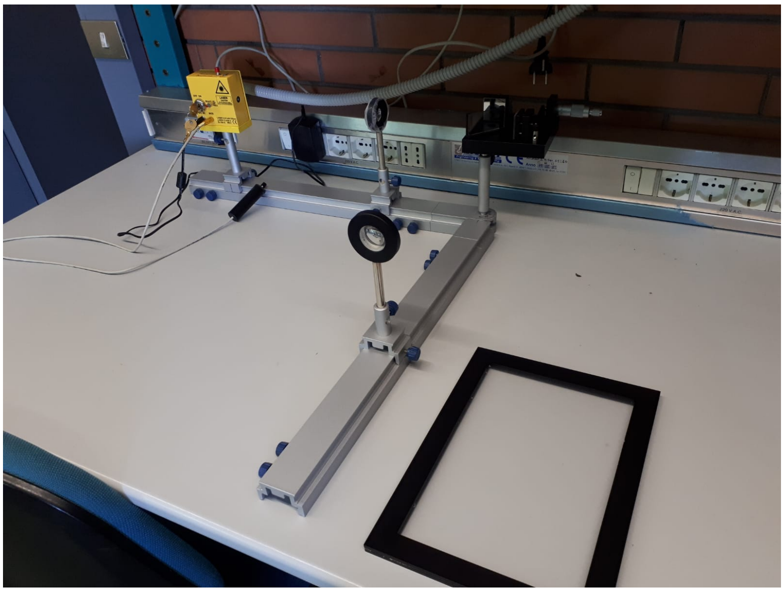


FIGURE 2.11: A Michelson interferometer with a laser as light source.

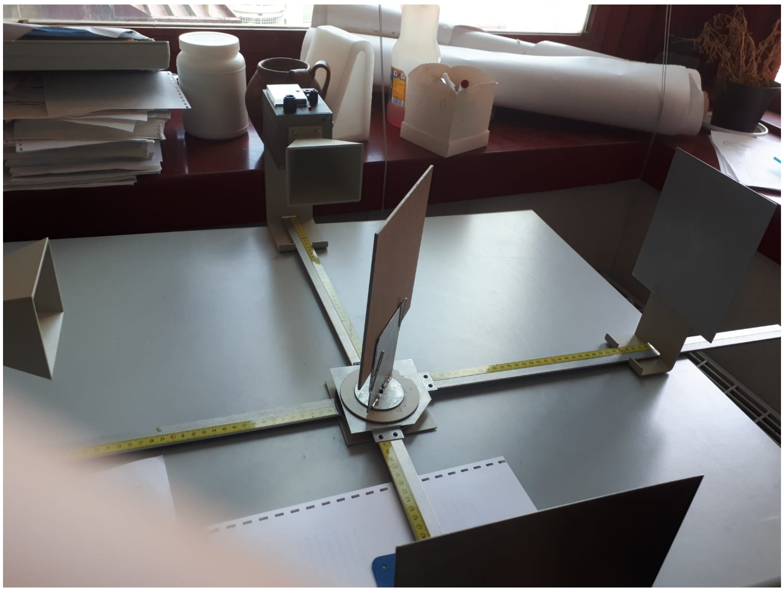


FIGURE 2.12: A Michelson interferometer with a microwave antenna a light source.

As shows figure 2.14, since the earth is moving we expect two different phases of the two beams when arriving on the screen. As the students already knew, this is not so. The interference figure does not change, even if we take the experiment at distance of six months.

To explain this result we cannot use the Galilean transformation, so the Lorentz transformation is proposed as a *correction*, or more precisely a generalization of the Galilean transformation.
In the next section, I will narrate how I proposed to interpret the Lorentz transformation

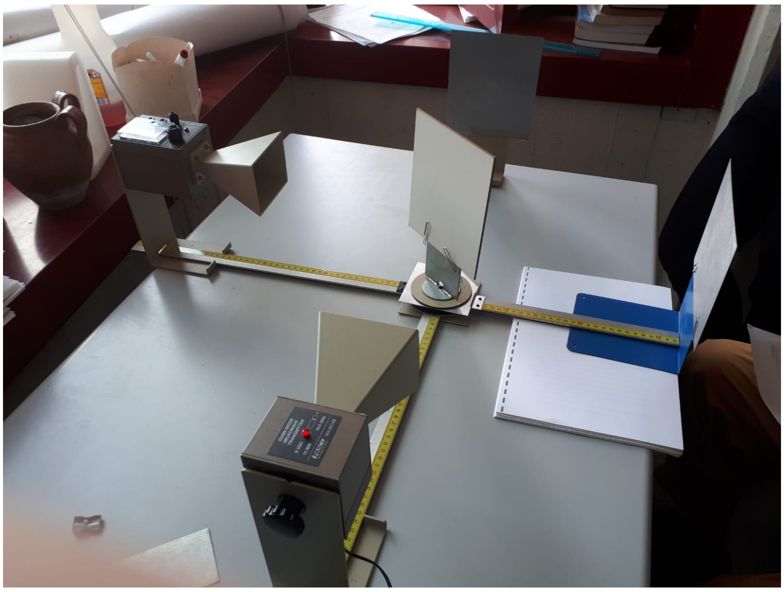


FIGURE 2.13: A Michelson interferometer with a microwave antenna a light source.

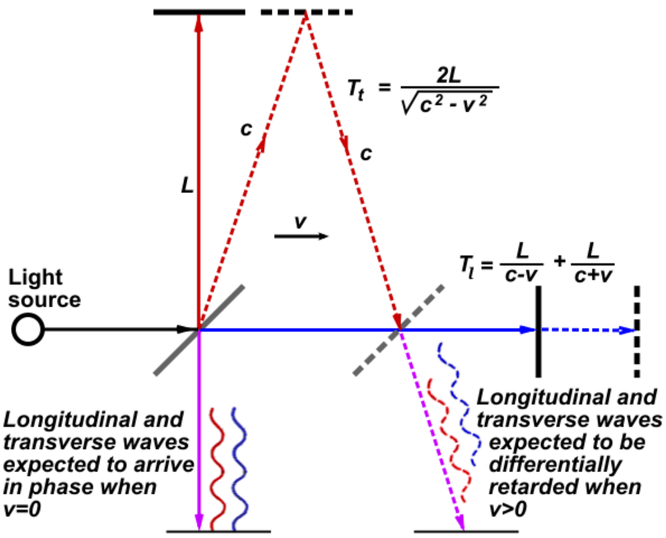


FIGURE 2.14: Expected differential phase shift between light traveling the longitudinal versus the transverse arms of the Michelson–Morley apparatus. Image taken from Wikipedia by the author Stigmatella aurantiaca.

with the students.

2.3 Lorentz transformations, thought experiments, four-vectors and relativistic mechanics

"We must admit with humility that, while number is purely a product of our minds, space has a reality outside our minds, so that we cannot completely prescribe its properties a priori. - *Carl Friedrich Gauss - Letter to Friedrich Wilhelm Bessel (1830)*"

"Time and Space ... It is not nature which imposes them upon us, it is we who impose them upon nature because we find them convenient. - *Henri Poincaré - The Value of Science(1905)*"

So, to take into account for the fact that the speed of light appears to be the same in every inertial reference system, we introduce the Lorentz transformation

$$x' = \frac{x - vt}{\sqrt{1 - \frac{V^2}{c^2}}}$$

$$t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \frac{V^2}{c^2}}}$$

where we have assumed that the movement is only in the x direction and c is the speed of light. Needless to say, if v is small compared to c this are the Galilean transformation, in this sense the Lorentz transformation generalize it. This transformation solves the problem since space is contracted exactly of the distance that the light beam should have travelled and the time is expanded of exactly the amount of time we expected. The tough experiment on the contraction of times, the simultaneity of the events and the length contraction can be found on any textbook which treats special relativity even at a high school level and the students were well prepared on this subject. An important feature which I emphasized of this transformation is that to obtain space or the time coordinate in the new reference system boosted with a speed V, we need both the space and the time and the space coordinate in the old system. This means that the two dimensions are mixed up, this is why the plane in which we represent the objects, the space-time diagrams, not only rotate but also stretches. This is also why we say that space and time form a unique structure. Moreover, I emphasized that physical quantities are defined by the measurement process. The contraction of times and the dilatation of lengths is what our measuring processes gives us back.

After this, instead of discussing all the possible paradoxes, I have drawn how a straight line in a space-time diagram, is transformed. While in a Galilean transformation all line a rotated, with a Lorentz the line that represents an object moving with the speed of light is fixed. For a mechanical visualization of this phenomena, we suggest to look at the video on the web page *https://www.youtube.com/watch?v=Rh0pYtQG5wI*, where a mechanical instrument that performs a Lorentz transformation in the space-time diagram has been realized. In simple words, the plane is rotated and starched, and this is to account for the constant of the speed of light in every inertial system of reference.

In analogy with the path followed in the first chapter, we can now build relativistic mechanics. The most effective way to do so is to find the quantity that preserves in time and that can be easily translated in a different reference frame. For the analogue of the momentum, it is simple to see that the quantity

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}} \tag{2.7}$$

where m is the mass and v the velocity of the object. I emphasized the subtle difference with the Lorentz transformation, here we have no V since there is no change in a second reference system with speed V with respect to the first.

With this new definition of momentum, every exercise previously solved in non-relativistic mechanics can be proposed again and solved again.

But we think that leveraging on the ideas of symmetry and conservation we can go a little further in the formalization of special relativity, so in the last part of my activity with the students, I introduced the four-vectors.

I argued that, since in classical mechanics we wanted objects that change in a specific way under Galilean transformation so that equation of mechanics are always valid we want to do the same here.

I argued that the distance in space is computed via Pythagoras's theorem and that in classical mechanics we assume to live all in the same instant of time. But we know from relativistic mechanics that it is not so, we need to add a time dimension to the object in the equation and we need to understand how to measure *distance* in the time axis. To do so, I argued as follow: we consider a light beam that *in vacuum* moves with speed

$$c = \frac{ds}{dt}. \tag{2.8}$$

In a different inertial reference system, times will be dilatated and lengths will be contracted, but the speed of light remains c

$$c' = \frac{ds'}{dt'} = c = \frac{ds}{dt}. \tag{2.9}$$

From the above equations follows that

$$cdt' - ds' = cdt - ds. \tag{2.10}$$

This means that the difference between the spatial distance ds and the time duration dt multiplied by the speed of light, is the same in every inertial reference system and we use this as the distance in space-time.
This little computation justifies the definition of vectors with four coordinates instead of three, one represents the time dimension and the other the spatial dimensions. When computing the modulus of this four-vector, we need to make the difference between the spatial and the time distance. This is the Minkowski metric, and we have defined the Minkowski space. If I had more time with the students I would have liked to emphasize that differently than in the usual Euclidean space, the modulus of a vector can be imaginary, and this leads to the definition of the light cone (which the students had already seen), the distinction of time-like and space-like four-vector and assures the validity of the causation principle. Instead, to trigger students thought for the last time in the activity, I asked them how to complete the momentum with a fourth *time representative* element. Arguing that we needed something that under transformation changes as the modulus of the momentum, since the difference with the modulus of the spatial and time coordinates must be unchanged. Remembering that the kinetic energy can also be written as $E = p^2/2m$, we arrived at the conclusion that this *time element* must have the dimension of an energy divided by c. With the definition of the four-momentum the circle is closed, we have built the relativistic mechanics in a way that allows students to solve problems. To end the activity, we tried to interpret the modulus of the four-moment as a physical quantity. Since it is always positive and it has the dimension of the square of a mass multiplied by the square of a distance, we assumed that

$$\frac{|E|^2}{c^2} - |p| = m^2c^2 \tag{2.11}$$

and this leads to

$$|E|^2 = m^2c^4 + |p|c^2 \tag{2.12}$$

which, for an object with no momentum reduces to

$$E = mc^2 \tag{2.13}$$

This equation is used in many application like the production of nuclear energy and since the time was finished the lesson ended.
After this, I would have done two things. The first is the paradox of the rotation wheel. Since in a moving object only the lengths which are parallel to the motion contracts, the circumference and the radius should change their ratio. This means that the *rigid body* cannot be used as an approximation for any body in the framework of special relativity. The last thing that I would have emphasized is the fact that the electromagnetism, as described by Maxwell equation, takes already into account special relativity.
If we have a moving particle, this will generate an electric and a magnetic field. But if we place ourselves in the reference system in which the charge is fixed, we will only measure an electric field. Anyway, the paradox is fixed if we think the electric and magnetic fields as a single electromagnetic field, where the magnetic and the electric components of the fields adjust under a Lorentz transformation so that the description remains coherent. To arrive at the transformation of the fields would be the satisfactory justification for the union of the two different fields.
With this, the circle is closed and the relativistic mechanics is developed as a generalization of the classical mechanics. From the Galilean transformation and the Euclidean space, we have built classical mechanics, from the Lorentz transformation and the Minkowski space the relativistic mechanics. Anyway, this is already done in schools with different strategies. In the next chapter, we will see how a physical theory can use an even more general geometry to build more general mechanics and an attempt to introduce General Relativity in high school is pursued.

Chapter 3

Approach to general relativity

> I foresee two possibilities. 1: coming face to face with herself 30 years older would put her into shock and she'd simply pass out. Or 2: the encounter could create a time paradox, the result of which could cause a chain reaction that would unravel the very fabric of the spacetime continuum and destroy the entire universe! Granted, that's a worst-case scenario. The destruction might in fact be very localised, limited to merely our own galaxy.
>
> Doctor Emmet Brown - Back to the future

In this chapter, an approach to general relativity for students of the high school is presented. In the first section, a laboratorial activity in school is described in which the basic features of spherical geometry are presented, in particular, Girard's theorem is proved. Continuing the description of the laboratorial activity, the second section documents my attempt to introduce tensors and to interpret Einstein field equation. In the last part of the second section, I propose to revisit the simulation of the motion of Mercury to take into account relativistic effects but this has not been tried in a classroom for a shortage in time. The last section has a more speculative imprint, I try to present the Schwarzschild metric as a solution of the Einstein field equation and to discuss it in terms that are not merely popular but understandable by students.

3.1 Geometry on a sphere, metric and curvature

> Experience guides us in this choice without forcing it upon us; it tells us not which is the truest geometry, but which is the most convenient.
>
> Henri Poincaré - Science and Hypothesis

Though the goal of this chapter is to introduce general relativity, we start with a section on spherical geometry, which is also the focus of my laboratorial activity with a class of the last year of the Liceo Scientifico Calamandrei. We have done so because, even if general relativity uses hyperbolic geometry and not elliptic, it is fundamental that students understand that we can do geometry on any kind of surface, and the globe is a perfect example to connects non-Euclidean geometry with personal experiences. This topic is usually treated as mathematics, so in literature, the approach starts from the discussion on the axioms and postulates of the Euclidean geometry, as is done in [21] and [22], but we will not follow that path. We will adopt a more *experimental* approach, starting from the things that can be measured. Of course, this is a great opportunity to take and, if time permits, a discussion on the foundation of geometry should be performed, but our goal is to introduce physics and doing so we treat mathematics as a tool. We hope that future studies of mathematics will fill the gap.
Differently than other two experience, this time there was only one classroom of sixteen students and we were in the laboratory of the school. I have taken six world globe as the one showed in 3.1 and 3.2 and the experience lasted 2 hours. Students had already studied General relativity in mandatory lessons from the book [23], so the aim of the preset laboratorial experience is to probe, recapitulate and strengthen the knowledge on non-Euclidean geometry and general relativity.

To every group was given a survey, for the first hour they where let free to do it by themselves, I had just a function of support for the inevitable questions and doubts on how to proceed. The first three questions were:

1) Measure the length at the equator on the world globe and transcribe it

Circumference = ... cm

2) Did the measure of the circumference has errors? would you know how to estimate it? would it have any sense to make an average of the measure of all group to get a better estimation?

FIGURE 3.1: The inflatable globe used during the laboratorial activity. The nominal diameter is of 40 centimetres, with a scale of 1:35 000 000.

3) Compute the radius of the globe and transcribe it.

Radius = ... cm.

Would you know how to estimate the error?

This questions posed the focus on the computation of the radius of the globe, approximated to a sphere, reversing the well-known formula $C = 2r\pi$. Due to the short duration of the activity, the error analysis has just been appointed but it has not been carried out. As stated from the teachers with which we have collaborated for this work, in Italy the error analysis is treated in the first year of high school but it is never used in the following years. Consequently, it is forgotten by students and it is seen as useless. The analysis of errors have to be done in the first year, but it has to be recalled in every subsequent laboratorial activity to show its purpose and meaning. In this experience, students conjectured various possible error source, like the non-perfect adherence of the

FIGURE 3.2: A label on the inflatable globe warns from the global warming caused by anthropomorphic activities.

metre on the globe, but could not treat them quantitatively.

The second part of the survey contained the following questions:

4) Take a spherical triangle with sides given by the meridians and the equator. What is the sum of the internal angles in grade and radiants?

Sum of internal angles in grade = ... Sum of internal angles in radiants = ...

5) Placing the transparent millimetre paper on the globe, draw the spherical triangle. Now measure the surface of the triangle.

Surface of the spherical triangle = ... cm2

would you know how to esteem the error?

6) Now divide the area measured in 5) for the angular excess (of how much the sum of the internal angles is bigger than 180 grades) expressed in radians and make the square root of the number obtained.

Results = ...

sono operazioni da fre dovranno darlgi loro senso!

7) What are the physical dimensions of the number calculated in point 6)? What have you computed?

Some picture of how to perform this measure is in 3.3 and 3.4, where it is shown how the millimetre paper can be made adhere to the bloge to trace the triangle, then the surface can be measured just counting the square in the spherical triangle.

FIGURE 3.3: A transparent graph paper is make to adhere (to the best) on the globe in order to retrace a spherical triangle.

As in the first part, students identified the main source of errors, how to consider the squares crossed by the perimeter of the spherical triangle. They just guessed when to

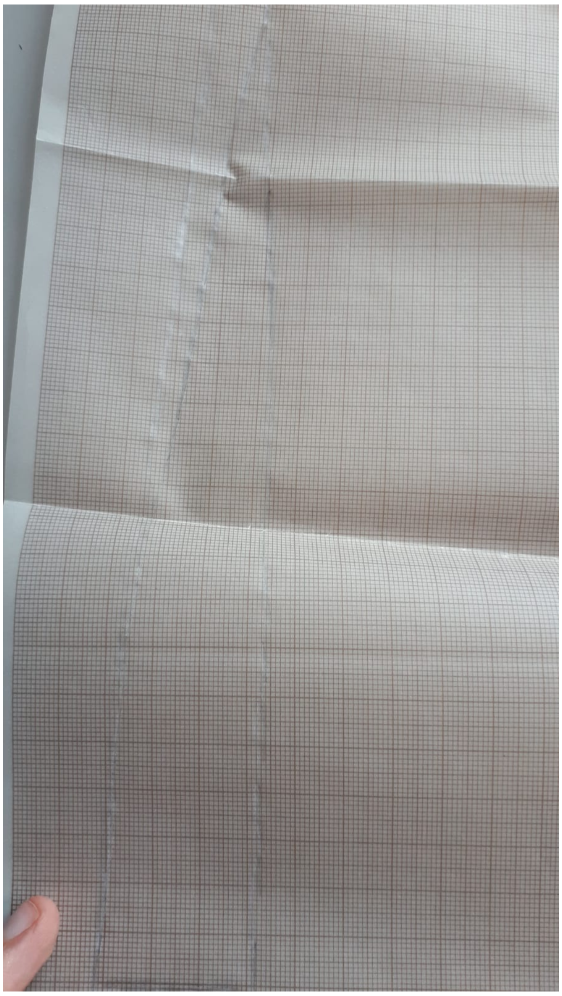


FIGURE 3.4: A figure obtained retracing a spherical triangle on graph paper.

consider just half square, but they could not take a quantitative trace of the errors. Furthermore, both for laziness and for a shortage of time, they did not counted the square millimeter but the square centimeter, the big square. Doing so, the error is easily seen to be of the order of $\pm 10cm^2$ or even more, since near the pole the meridians are not drawn on the globe and those part of the spherical triangle has to be guessed. As we will show in a minute, is to much for what we wanted to obtain.
The goal of this question is to introduce Girard's theorem, that connects the angular excess, the area of the spherical triangole and the radius of the globe. From the computation performed in question 6), they should have obtained the radius of the sphere, which is of the order of $20cm$ (it depends on how much the globe was filled). When I tried this experiment counting the square millimeters, the two estimations of the radius where consistent and they where almost equal so that even without doing error analysis I imagined students would have guessed that they had obtained the radius once again. It was not so, and this is a feature to correct in eventual future experiences.
Girard's theorem has been proved in the second part of the activity, after all students

completed the survey. Before that, there was only the last question:

8) What did you think is the shortest pattern between Rome and New York? try to find it on the globe. It is what you expected? If it is not, what is the difference?

As students already knew, the shortest distance is not find following the parallel (Napoli and New york follow approximately on the same parallel) but following the maximum circle that contains the two city, as shown in figure 3.5 and 3.6

FIGURE 3.5: If we measure the distance between New York and Napoli following a parallel the meter did not adheres to the globe and we have a bib efficient route.

FIGURE 3.6: If we measure the distance between New York and Napoli following the maximum circle on the globe that contains both cities, the meter will adhere perfectly on the globe and marking the most efficient route.

Once every group finished the survey we had a short break, the activity restarted with a discussion of the survey.
There first three questions posed no problem to the students, except for the discussion on the error analysis aforementioned. The second part of the survey had risen one big problem, the interpretation of the quantity computed in question 6). Some students arrived at the conclusions that it had the dimensions of a length divided by the square root of radians, which in turn posed a question of the *dimension* of radiants Of course, the radians have no dimension, since they are defined as the ratio of lengths, as was remembered by the students. Anyway, the quantity computed should have been the radius of the globe, but the experiment failed for the aforementioned reasons, so to convince students that that was actually an alternative way to measure the radius I proved Girard's theorem.

Girard's theorem states that, given a spherical triangle of area $A(T)$ and angular excess $\xi(T)$ on a sphere of radius R,

$$A(T) = R^2\xi. \tag{3.1}$$

The proof that we propose follows closely the proof given by *professor John Polking* at http://math.rice.edu/ pcmi/sphere/gos4.html and the images 3.7, 3.8 and 3.9 are taken from https://orfe.princeton.edu/ rvdb/WebGL/GirardThmProof.html, that also propose the demonstration of professor Polking, but completes it with the applet needed to manipulate the sphere.

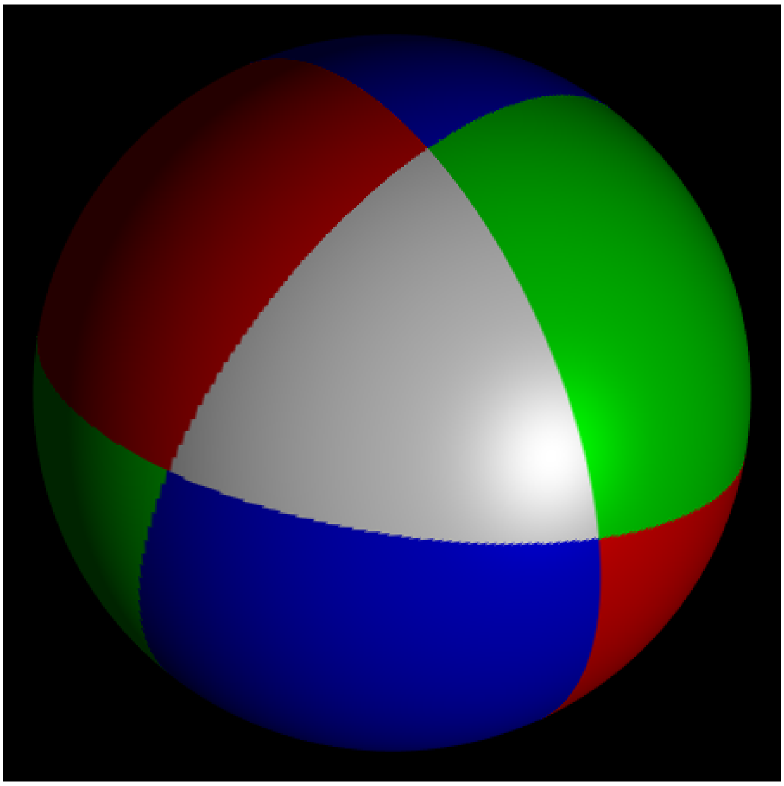


FIGURE 3.7: Three maximum circle on a sphere defines eight triangles, four in the *upper half* and four in the *lower half*. For symmetry reason, every triangle has congruent triangle on the other side, in this figure congruent triangles have the same colour.

To prove Girard's theorem, we must first derive the formula for the area of a *lune* of a sphere, the portion of a sphere between two maximum circles. Also for this, we follow professor Polking, that in $http://math.rice.edu/\ pcmi/sphere/gos3.html\#1$ show how this can be done. Since the surface A of all the sphere is $A = 4\pi R^2$, if we divide the sphere with two maximum circles in four equal parts, the four lunes will have right angles of $\pi/2$. With a proportion, we can approximate that given a lune whose lunar angle is α its area $A(l)$ is equal to

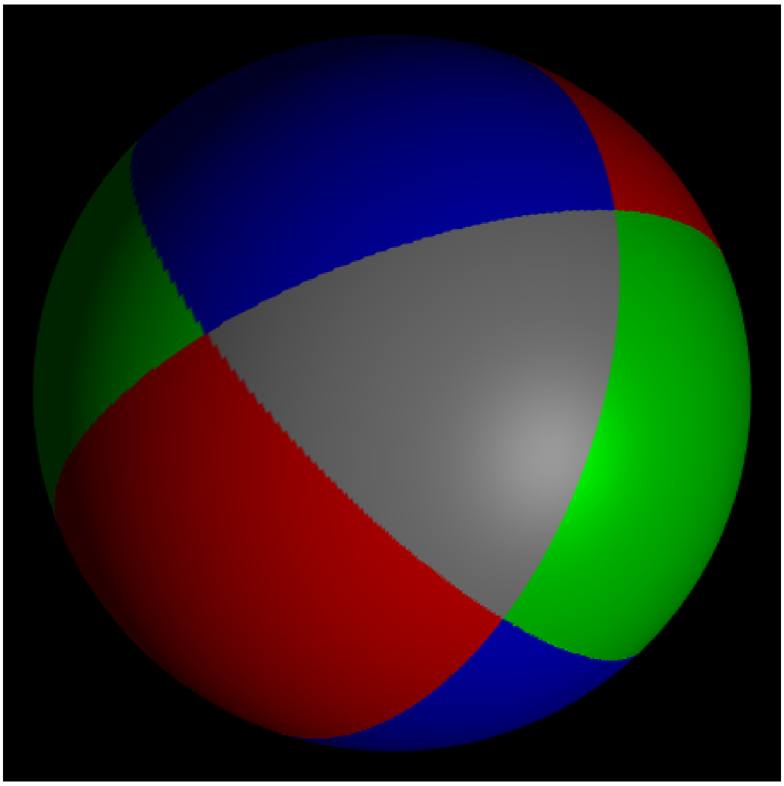


FIGURE 3.8: The other half of the sphere, where the opposite of the withe triangle has been coloured in grey.

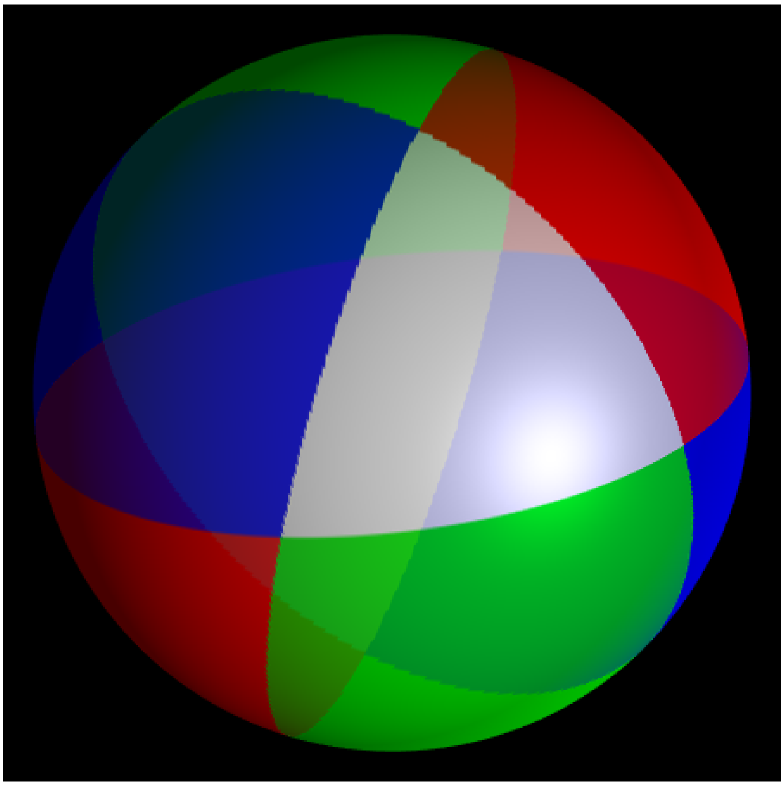


FIGURE 3.9: With this apple is also possible to rotate the sphere, zooming it or making it translucent, as in this image. This helps while counting all the lunes during the demonstration of the Girard's theorem.

$$A(l) = R^2\alpha. \tag{3.2}$$

If now we look at the image the images of the sphere from the applet, we see that the sphere is divided into eight spherical triangles by three maximum circles. Since angles opposite to a vertex are equal, it follows that triangles of the same colour, who are situated opposite one with respect to the other on the sphere, are congruent. In particular, the triangle opposite to the white one is coloured in grey, since we focus on finding the area of the white triangle. To do so, we take into account both lunes that contain the green triangle $A(l_g) + A(l_{g'} = 2A(l_g)$, both lunes that contains blue triangle $A(l_b) + A(l_{b'} = 2A(l_b)$ and both lunes that contains red triangle $A(l_r) + A(l_{r'} = 2A(l_r)$. In doing so we have covered all the surface of the sphere, plus we have taken into account two times the white triangle and two times the grey triangle. To sum up:

$$2A(l_g) + 2A(l_b) + 2A(l_r) = 4\pi R^2 + 4A(T). \tag{3.3}$$

We can now use the formula for the area of a lune previously derived, we will mark the angles as lunar angles g, b and r, obtaining

$$4\pi g + 4\pi b + 4\pi r = 4\pi R^2 + 4A(T) \tag{3.4}$$

$$A(T) = R^2(r + p + g - \pi) \tag{3.5}$$

but g, b and r are the internal angles of the white triangles, so the term in parenthesis is the angular excess and the theorem is proved.
Every ingredient of this theorem can be understood from high school students and it is not less elementary than many theorems on Euclidean geometry usually discussed. This is also the proof that non-Euclidean geometry can be treated in a quantitative and rigorous way in high school.
Now, we can ask if we can draw a triangle big enough on the ground to measure an angle different from π. Of course, we imagine a ground always parallel to the plumb line. To get an angular excess of one degree, which are equal to $\pi/180$ on a sphere with a radius $6371km$, which is the approximated radius of the Earth, the area of the triangle have to be $A(T) = 6371^2(\pi/180) \approx 708402km^2$ which is more than twice the surface of Italy! Conversely, for a triangle with a surface, let us say of 10 square meters, in a laboratory, the angular excess would be of $A(t)/R^2 \approx 4 \times 10^{-7} rads \approx 2 \times 10^{-5} degrees$. This means that if the radius of the sphere is big with respect to the dimensions of the triangle, it

seems to be flat with good approximation. To see that the Earth is not flat, one has to seek for other kinds of proofs, like the ships that rise on the horizon.

Once the theorem has been proved, and the students involved had no problem following all the steps of the proof, one can use it in reverse as a way to define curvature as the inverse of the radious. As I showed in the laboratory without going into the detail of the computations, for many surfaces we can approximate a little piece of it with a sphere of radius R, but for different little pieces, who are actually points, we need a sphere of different radius, moreover it could change *side* and we can have positive or negative curvature. Indeed a surface has negative curvature at a point if the surface curves away from the tangent plane in two different directions and the classic example is a saddle.

Before going into discussing how to apply non-Euclidean geometry to physics, we discussed the last question of the survey and how it is related the previous questions.

I used question 8) to introduce the concept that we need to introduce a way to measure a *distance*. We agree that the distance is the *shortest* path between two points. Of course, we can get from one point to another following infinite paths, but the distance is only the shortest one (or ones if there are two or more path with the same shortest length).

On the globe, if one goes from Napoli to New York following the maximum circle instead than the parallel, the route is shortest of approximately a centimetre. Anyway, map on the globe is in scale $1 : 35000000$, so an aeroplane would save approximately $350km$!

Anyway, on a globe with a different radius, the distance is measured differently. And if the surface has a variable curvature, the distance must be measured differently in every little piece.

The connection between the curvature and the way in which we measure distance is investigated in the next section. At this point, the last half hour of activity, become a frontal lesson where I just sought interaction with the students to help them follow the line of thought.

3.2 Tensors and Einstein field equation

Dare un calcio alla ragione e fare posto all'impossibile: è la nostra filosofia, no? È così che siamo arrivati in superficie. E allora perché non dovremmo riuscire ad andare sulla Luna?

Kamina - Gurren Lagann

In the previous section, we have seen that Euclidean geometry, that for our particular goals means to measure distance with the Pythagorean theorem and that the sum of the internal angles of a triangle is π, did not describe well *all* surfaces. There are surfaces that are described with a different geometry, and a discussion on the fifth postulate would be well placed at this point. But I wanted to discuss general relativity, so I focused on another question: Are there non-Euclidean spaces in which distance is not measured with a straight line?
The question is reasonable, indeed Gauss and Riemann, two great mathematicians and physician of the nineteen century, approached this problem trying to measure the angular excess of big triangles. They tried with a triangle whose vertex were approximated with the position of three mountains tops, but the angular excess was consistent with zero within the experimental errors. The same happens if one tries to measure the angular excess of a triangle whose vertex were approximated with the position of three stars. It is important to note that we are not talking of triangle drawn on a surface, we are talking of a triangle defined by three points in space, so if the angular excess is different from zero space itself cannot be described with Euclidean geometry. It seems then than experimental results suggests that space is well described by Euclidean geometry, but as we have seen in the previous section the angular excess is very little, but we can try to highlit it in other ways. As we will see in the next section, the motion of the planet Mercury, that we have simulated in the first chapter, differs from the one predicted by Newton theory, even if we add the contribution of all the planets. As reported in [24], Mercury's perihelion makes one complete rotation around the sun in approximately 230000 earth years. This means that in a century it moves on the celestial sphere of 566 arcseconds, where $1 arcsecond = 1/3600$ degree. The other planets (most of all Jupyter) contribute for only 523 arcseconds per century, so there is a disagreement of 43 arcseconds per century between the Newtonian theory of gravitation and the actual motion of Mercury. A simple solution for this disagreement would have been the presence of unseen massive bodies near mercury, in analogy with the discovery of Neptune from the discrepancies on the theorized orbit of Uranus. But this was not the case, to explain the discrepancy

between the concept of gravity itself had to be rethought, and general relativity had to be formulated by Albert Einstein.

In this introduction to General relativity, we will follow [25] but we will try to avoid mathematics that is not known to a high school student that is not introduced within this text.

The first thing i emphasized with the students, is that in classical and relativistic physics we used respectively vector and four-vectors to describe quantities. This mathematical object has the the propriety that if a relation is between vectors is valid in a given reference frame, it will be valid in every reference frame who moves with uniform velocity with respect to it. In other words, if $\vec{F} = m\vec{a}$ is true in a given reference systems it will be true in every reference systems that moves with constant speed with respect to it. The change to a different reference systems with uniform velocity is called *Galilean transformation* and classical mechanics is *invariant* under Galilean transformation. When generalizing to the relativistic mechanics, we referred to Lorentz transformation instead of Galilean transformation. This time we want to generalize the theory making it invariant for an even more abstract transformation, called *diffeomorphism*.

The definition of diffeomorphism is that given two manifolds M and N, a differentiable map $f : M \to N$ is called a diffeomorphism if it is a bijection and its inverse $f^{-1} : N \to M$ is differentiable as well. Anyway, I have not given to the students the rigorous definition of diffeomorphism, to do that I should have first to define what is a *manifold* and to generalize the concept of differentiation from ordinary functions. Instead I focused on the geometric idea of chart and coordinate systems. On the globe we have attached the millimeter paper, which is a representation of a coordinate system. More precisely, we have attacked a Cartesian coordinate system, with two orthogonal axes, to a portion of the sphere. On the globe there is a different coordinate system, represented by parallel and meridians. It is called *spherical coordinates* and longitude and latitude are the coordinates.

But we can think of more general diffeomorphism, it suffices that there are no cut or discontinuities, for example the one in figure 3.10.

If I had more time at my disposal, this would have been the occasion to connect physics and art. For example, in figure 3.11 a fish image is distorted when the greed on which is drawn down is distorted. Indeed the artist Maurits Cornelis Escher, played with geometry and perspective in his works, making many artistic representations of mathematical models like 3.12 which is a representation of the *Poincarè disk model*, where a hyperbolic geometry is represented inside a disk. On the Poincarè disk, the *straight lines* are circles, so the *fifth postulate* or *parallel postulate* that states that in a plane, given a line and a point not on it, at most one line parallel to the given line can be

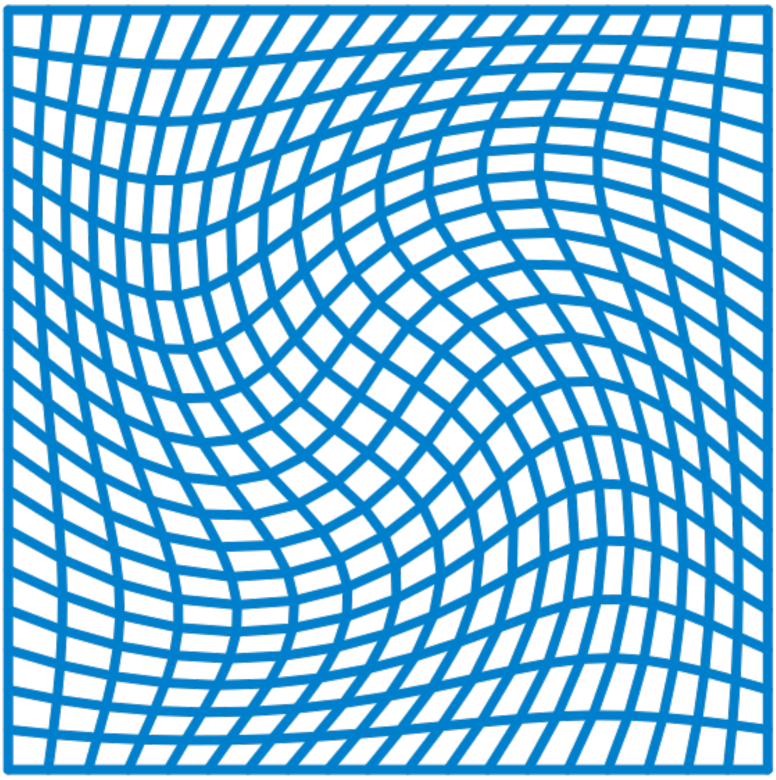


FIGURE 3.10: The image of a rectangular grid on a square under a diffeomorphism from the square onto itself - Image taken on wikipedia - Author: Oleg Alexandrov

drawn through the point do not hold. Moreover, as we approach the circumference the distances are shortened. To deepen this topic we suggest, aside from the technical literature, to look for *geogebra* (a didactic software) simulations of the disk like the one on www.geogebra.org/m/R5e9AggU.
Also, to deepen the connection between science and art we suggest [26], where the connection between geometry, painting and informatics is made as clear as possible.

Anyway, I had not the time to go into this detail. Instead, since they had always seen it in the mandatory lessons, I passed to the interpretation of the Einstein field equation. As I stated before, these equations need to relate object that, during a diffeomorphism, change in a way that the relation still holds. Since a diffeomorphism may deform the space in a different way along different directions, this objects and the way it is transformed must take this into account. Of course, we are talking about tensors which I introduced just a table of numbers like sometimes is done with matrices.
A good way yo introduce tensors is in the video at www.youtube.com/watch?v=f5liqUk0ZTw, where it is introduced with children toys as a generalization of the three-dimensional vectors. In simple and elementary words, it is described the stress tensor, we have three components (the diagonal of the table) which represent the stress along three spatial directions, while the other one are *spurious* components that take into account how the

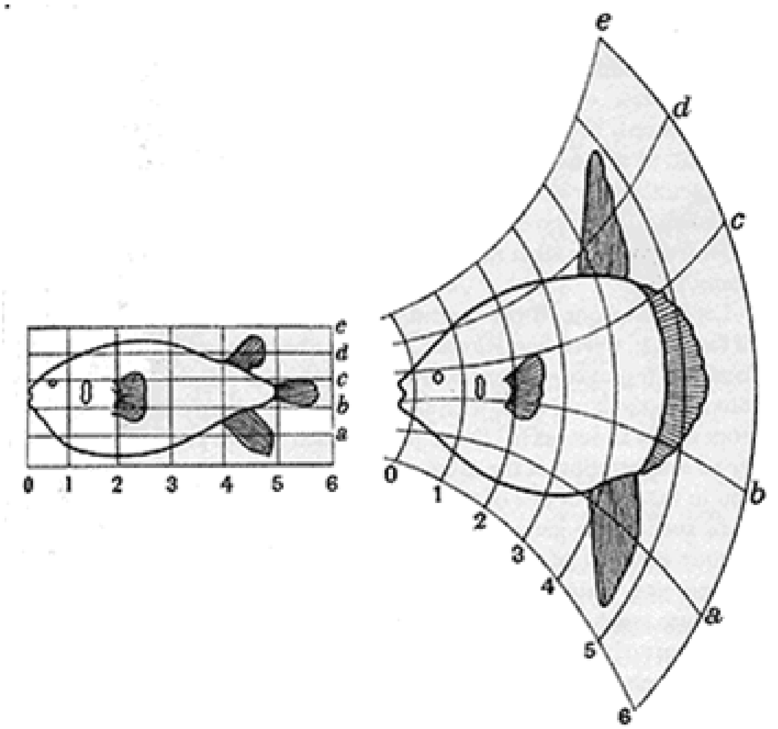


FIGURE 3.11: How the image of a fish is distorted when a diffeomorphism is applied to the rectangular grid in which is drawn down.

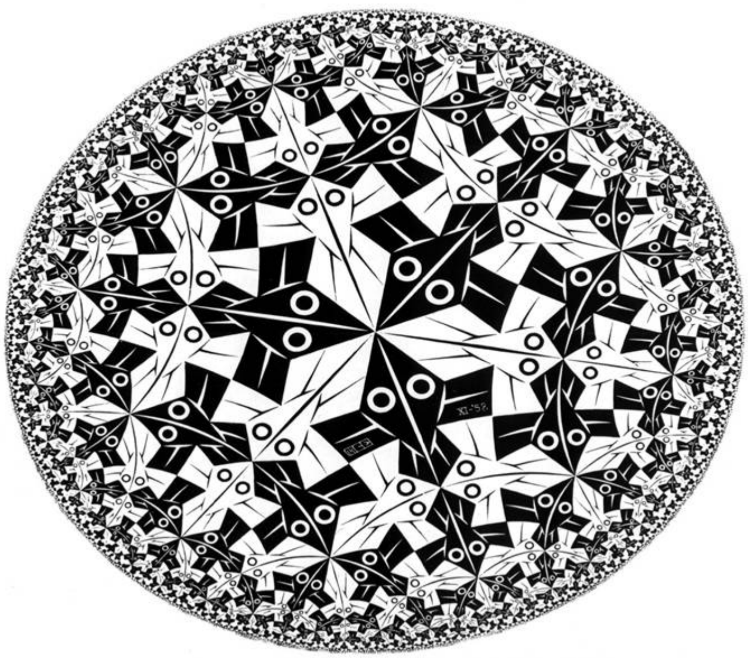


FIGURE 3.12: A famous paint of M.C. Escher. Approaching the end of the circle, distances are shortened. This is also a representation of an hyperbolic geometry where all the space is represented on the disk.

stress is projected into different directions. Of course, this has to be generalized to tensors in Minkowski space. Moreover, the video does not shows quantitatively how a tensors transform under a change in coordinates. An attempt in doing so can be found on the video on www.youtube.com/watch?v=CliW7kSxxWU, where a reasonable definition of *contravariant* and *covariant* components (if the given components changes following the same rule of the basis or it inverse) is given.
To treat all the topics I talked in this section until now would be, in our opinion, a very good starting point to interpret the Einstein field equation, that we can give without derivation as

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu} \tag{3.6}$$

At this point, I investigated the equation term by term, looking for what students already knew and filling some gap.
The index runs from 0, the time component to 3, the spatial components as in analogy with the four-vectors in special relativity. $g_{\mu\nu}$ is called *the metric tensors* and gives information about how to compute a distance in the given space. The example I presented where the one of the standard three-dimensional Euclidean space, where the metric is:

$$\eta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

So, given a point $x_i = (x_1; x_2; x_3)$ we can compute the square of its distance from the origin as $x_i\eta_{ij}x_j$ where I preferred not to use upper index since I had not explained the difference covariant and contravariant components. Students also did not know the algorithm for the multiplication of matrices, so I just make them note that they already knew the results as the sum of the square of the components:

$$d^2 = x_1^2 + x_2^2 + x_3^2 \tag{3.8}$$

As a second example, I proposed the Mikowski metric:

$$g_{\mu\nu} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.9}$$

and pointed out that the square of the distance is computed as:

$$d^2 = +x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{3.10}$$

In other words, the metric tensor codifies the way to measure distances in the space, and by consequence its geometry. As we have seen in the previous section, this must be connected to the curvature. The tensors that codify the information about the curvature is $R_{\mu\nu}$, and R is the sum of the elements of the diagonal of $R_{\mu\nu}$.
On the other side of the equation, G is Newton gravitational constant, c is the speed of light and $T_{\mu\nu}$ is the generalization of the stress-energy tensor that codifies the information about the mass and energy in the space-time region we are considering.
At this point, my time was over, and I ended the lesson stating that we can interpret this formula seeing the gravitation not as a force, like in Newton theory, but as a geometry in space-time. Mass cures space-time and the straight line is not the shortest distance anymore, so object follows the *geodesic* (line of shortest distance) which now is a curve. In General relativity, we don't seek for the forces that act upon the objects changing their state of motion, the object continues persevering in their motion but the geometry of the space-time changes. So to find how an object moves we need to know how to compute the geodesic in a geometry where the metric and the curvature are given.
Unfortunately, I had no time enough to talk about the geodesic equation, that can be used to compute the motion of the particles in this new framework.
To talk about the geodesic equation I would have needed two more ingredients, the *four velocity* and the *Christoffel Christoffel symbols*. The four-velocity can be seen as a generalization of velocity in three-dimensional space to the space-time. We start from the four position of a point, $X = (cx_0; x_1; x_2; x_3)$ We note that the first coordinate cx_0 is the time coordinate, but this time we cannot divide the space travelled by the time lapsed, or else we would not obtain a four-vector. Anyway, the trajectory of the object in the space-time, also called world-line, will have a tangent in every point, and that is the four-velocity. From an algebraic point of view, we need to find a parameter that has the function of the time in the classical theories. For now, we shall use the proper time τ. In this notation, the four-velocity U^μ is defined as:

$$U^\mu = \frac{dX}{d\tau}. \tag{3.11}$$

Unfortunately, an introduction of the Christoffel symbols without making use of differential geometry is less simple. I would just say that we need to define the concept of derivative on spaces that are not Euclidean. The derivative is indeed connected to how we measure distances and how we make limits, so if the coordinate systems are not the

usual one, the Cartesian, the derivative has to be revised. To do so, we introduce the Christoffel symbol $\Gamma^{\mu}_{\alpha\beta}$.
We can now write the geodesic equation, which we shall use like the equation of motion, as:

$$\frac{U^{\mu}}{d\tau} = \Gamma^{\mu}_{\alpha\beta} U^{\alpha} U^{\beta} \tag{3.12}$$

The connection from the Einstein field equation and the geodesic equation is in the Christoffel symbol, the Einstein equations tell us the geometry of the space and the geodesic equation tells us the trajectory of the bodies. Unfortunately, the Einstein field equation is hard to solve due to the non-linearity of the equations. This is a problem that can arise also in classical physics when applying $F = ma$. If we know the force we derive the acceleration or vice versa, but what if the force depends upon the acceleration? Another classic example is a particle with charge q moving in an electric field E, where the field generated by the charge is often neglected. So in the Einstein field equation, the geometry tells the mass how to distribute in the space. But on the right side of the equation, the mass and energy distribution tells the space-time how to *curve* so to define the geometry.
There are two strategies that we can adapt to solve the Einstein field equation then, one is to use numerical methods. In particular, we would like to cite *Regge Calculus* [27] where the manifold is replaced with a polyhedron with an appropriate number of dimensions. This approach also has the advantage of not needing any coordinate systems to work, indeed if the theory is invariant under diffeomorphism it means that is only helpful but not needful to introduce one!
The other approach consists in studying particular cases in which the equation is simple. For example, if the space-time is flat and we consider a particle whose mass is negligible, the Christoffel symbols are all zeroes. This means that the variation in the proper time of the four-velocity is zero, in other words, the four-velocity is constant. So a *free particle* in a flat space moves with uniform four-velocity.
In the next section, we will see another situation in which the Einstein equation can be solved, obtaining the Schwarzschild metric.

3.3 Schwarzschild metric

Doc: Marty, you gotta came back with me!
Marty: Where?
Doc: Back to the future!

Back to the future

One significant solution of the Einstein field equations can be obtained assuming a spherical symmetry for the systems. This is a good approximation for the solar systems, but also for more exotic systems like the black holes.
We remember that, in space-time, the spherical coordinates are:

$$\begin{aligned} t &= t \\ x &= r \sin\theta \cos\phi \\ y &= r \sin\theta \sin\phi \\ z &= r \cos\theta. \end{aligned}$$

Given this coordinates, if we try to solve the einstein equation in the vacuum, with zero on the right hand, we obtain the Schwarzschild metric:

$$d^2s = +\Big(1 - \frac{2Gm}{r}\Big)dt^2 - \Big(1 - \frac{2Gm}{r}\Big)dr^2 - dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2). \tag{3.13}$$

We have assumed at this point that the notion of the differential is known to students and i would not suggest an actual derivation of the metrics, but to verify that it is a solution plugging it into the Einstein field equation. We can use the Schwarzschild to find the trajectory of a particle that moves in a radial field and this is a good model for a planet spinning around the sun. Unfortunately, we have not found a way to derive, from Schwarzschild metrics and the geodesic equation, the equation of motion without mathematics that is not usually treated in high school, but the formula ca be presented as a *correction* to the law of universal gravitation:

$$F \approx \frac{Gm_1m_2}{r^2}\big(1 + \frac{\alpha}{r^2}\big). \tag{3.14}$$

The $\approx$ is necessary since we have omitted a few terms in the complete equation that takes into account also the chance of massless particles.

For a derivation of the formula of the precession of the perihelion, we suggest [28], [29] and [30]. In particular, the last one derives it with only algebraic calculations.
For our goals, the only difference from the classical formula is the the new term proportional to $\frac{\alpha}{r^2}$. In [24] a way to numerically measure α for the planet Mercury based on the astronomical data is presented, and it is of the order of $2,4 \times 10^{14)} m^2$. The idea is not to simulate 230000 earth years of motion but to measure on a more suitable interval on time and then to apply a linear regression. We propose something more simple, we just modify the program of chapter one to see what happens to the planet trajectory when the new term is adjoint to the classical formula. To make the effect visible, we have assumed $\alpha = 10^(31)$ as can be checked on the code:

```
from math import sqrt,atan, pi
from matplotlib.pyplot import *
from ipywidgets import interact
from scipy import stats
%matplotlib inline

def mercury_GR(N = 170):


    t = [0]
    Dt = 76005.216

    G = 6.67408*10**-11
    M = 1988500*10**24
    m = 0.33011*10**24

#To set initial condition we referred to the NASA fact sheet,
#https://nssdc.gsfc.nasa.gov/planetary/factsheet/mercuryfact.html
#The simulation starts with the plant in his aphelion.

    alpha = 10**31

    x = [69.82*10**9]
    y = [0]

    vx = [0]
    vy = [38.86*10**3]

    r = [sqrt(x[0]**2 + y[0]**2)]
    v = [sqrt(vx[0]**2 + vy[0]**2)]


    ax = [-G*M*x[0]/r[0]**3 - G*M*x[0]*alpha/(r[0]**6)]
    ay = [-G*M*y[0]/r[0]**3 - G *M*y[0]*alpha/(r[0]**6)]


    for i in range (1, N):

        x.append(x[i-1] + vx[i-1] * Dt + 0.5 * ax[i-1] * Dt**2)
        y.append(y[i-1] + vy[i-1] * Dt + 0.5 * ay[i-1] * Dt**2)
```

```
        r.append(sqrt(x[i]**2+y[i]**2))

        ax.append(-G*M*x[i]/r[i]**3 - G*M*x[i]*alpha/(r[i]**6))
        ay.append(-G*M*y[i]/r[i]**3 - G *M*y[i]*alpha/(r[i]**6))

        vx.append(vx[i-1] + 0.5 * (ax[i-1] + ax[i]) * Dt)
        vy.append(vy[i-1] + 0.5 * (ay[i-1] + ay[i]) * Dt)

        v.append(sqrt(vx[i]**2+vy[i]**2))

        t.append(t[i-1] + Dt)


        #Plot the results

    figure()
    plot(x, y)
    title("Trajectory, both axes have meters as units")
    grid()

    perihelion_x = []
    perihelion_y = []

    for i in range (1,N-1):
        if r[i] < r[i-1] and r[i] < r[i+1]:
            perihelion_x.append(x[i])
            perihelion_y.append(y[i])

    figure()
    scatter(perihelion_x, perihelion_y)
    title("Mercury perihelion motion, both axes have meters as units")
    grid()

    show()

interact (mercury_GR, N = (1,100000,1))
```

The codes produce two graphs, 3.13 shows how the trajectory is not an ellipse anymore. At any revolution around the sun, the perihelion moves, and in 3.14 is shown the position of it at any revolution.

This is the explanation for the precession of Mercury perihelion of which we talked at the beginning of this chapter. This shows that general relativity is not *just a theory*, it is a theory that people use to build models that describe the world around us.

The other example often made to emphasize the impact of general relativity on our society is the geolocation. Without the correction of the special and general relativity, it would be impossible to implement a geolocation. This would not only impact our everyday route to work but also the freight transport, geolocation is actually one of the

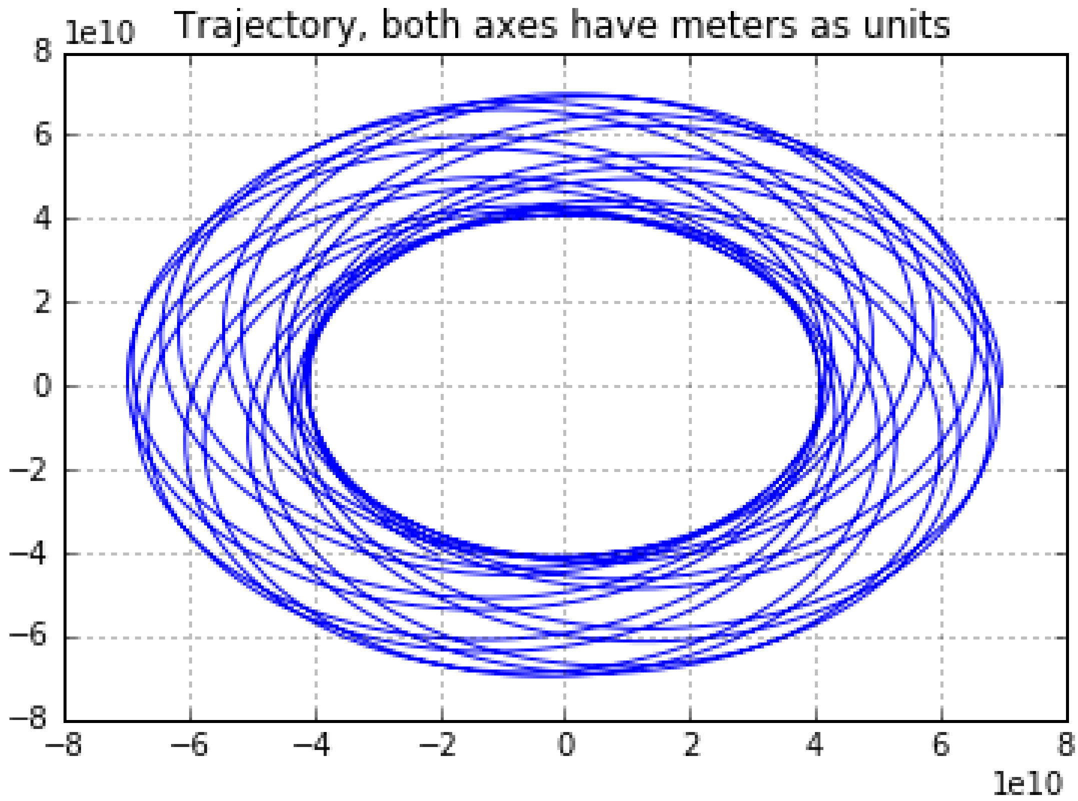


FIGURE 3.13: The trajectory of Mercury is not an ellipse anymore.

technologies that *shortened* the distances of our world in the last century.
Anyway, can also use Schwarzschild to talk about more mundane things, like *black holes*. When an object is dense enough, it is possible to get at a distance of $r = 2Gm$ from it. From Schwarzschild metric, we see that as we approach this distance, the distance in the dt component goes to zero but the distance in the dr component goes to infinity. This means that even light could not escape the gravitational field at this distance. In other words, as we approach this distance, also called event horizon, space stretches so much that we can never actually reach it.
This should give correct, even if not rigorous, understanding of general relativity to high school students. When I started to think on how to introduce general relativity in high school, I tried to take inspiration from classical mechanics, which is introduced without differential equation or Hamilton formalism. I hope this approach takes a step in that direction.
After this, many thinks could be discussed with students at a popular level but without falling into *fanfiction* slogans, like the cosmological constant Λ that can be added to the Einstein field equation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}. \qquad (3.15)$$

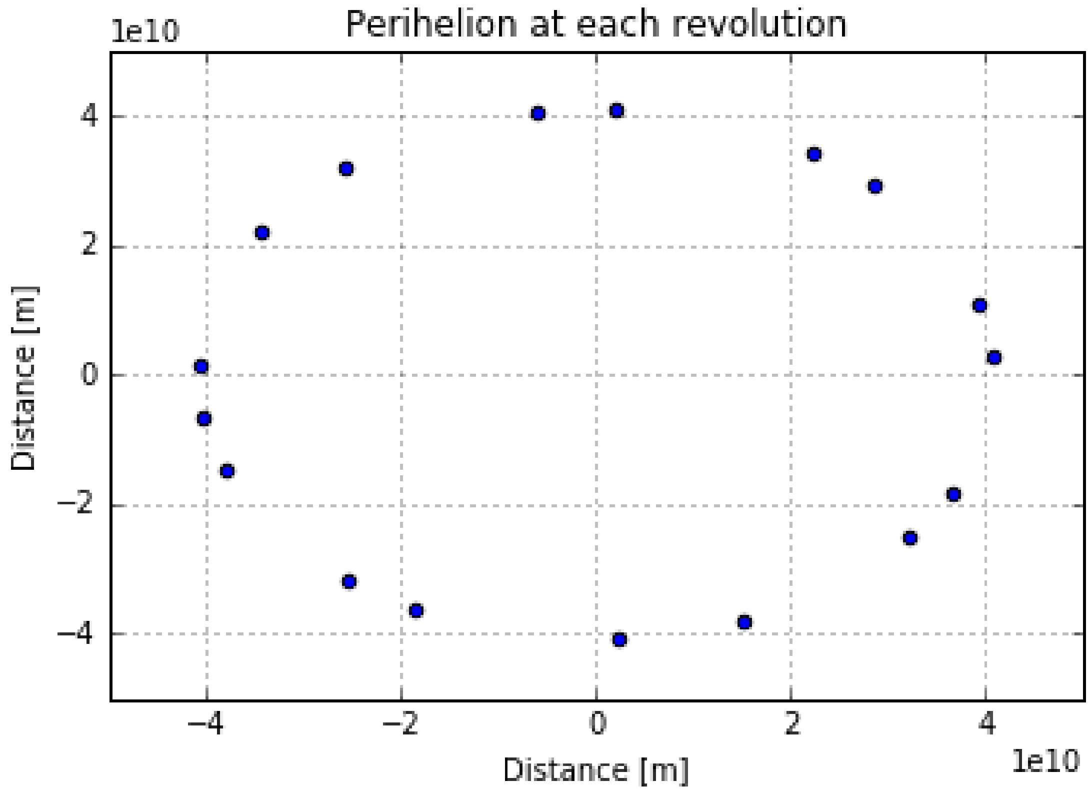


FIGURE 3.14: The position of the perihelion revolution after revolution around the sun.

A good book to continue the study of *modern* physics, once completed what we have proposed, could be [31], who ventures in some of the more speculative topics in physics like the search for a theory of everything.

Conclusions

Both teachers and students have enjoyed the laboratorial activities. As teachers pointed out, the most fruitful aspect~~s~~ was the enlightenment of new connections between topics. The focus on the crosscutting concepts like conservation ~~rules~~, transformations and symmetries allows a bridge between classical and modern physics. This is essential to promote a sound and profound understanding of physical theories, integrating different physics topics without renouncing to point out the ~~alterations~~ of meanings as we move from one context to another.

I have presented four laboratorial activities to test a new approach in the ~~building~~ of the mathematics and physics ~~curriculum~~ in high-school. ~~One activity~~ was on classical mechanics. Starting from experiments with a motion detector I discussed Galilean transformations as ~~rotation~~ in ~~the space-time diagram~~ and ~~the~~ momentum conservation. Emphasizing momentum conservation I connected topics that are usually treated ~~apart,~~ like Newton's laws and impulse. This approach leads to an easy generalization ~~to~~ relativistic mechanics ~~where Lorentz transformations can be presented geometrically.~~ This activity was proposed to two classes ~~joined~~, more than forty students, in the laboratory of the *Liceo Scientifico Statale Filippo Silvestri* with the collaboration of ~~the~~ teachers *Ilaria Limoncelli* and *Margherita D'Urzo*.

The second and third activities were presented to three classes ~~joined~~, more than fifty students, of the *Liceo Scientifico Statale Carlo Urbani* with the collaboration of ~~the~~ teachers *Maria Rosaria Camarda*, *Maria Loffredo* and *Annette Luongo*. ~~In~~ these activities I discussed electromagnetic waves, comparing them with mechanical waves, and special relativity. Students ~~were particularly pleased to see, with~~ instruments like the slinky and the microwave optic bench, phenomena they ~~only~~ discussed theoretically.

In the fourth activity, I presented non-Euclidean geometry in a class, less than twenty students, of the *Liceo Scientifico Statale Calamandrei* with the collaboration of ~~the~~ teachers *Marina De Cesare* and *Chiara Tarallo*. With the help of globes and an applet, I ~~proved~~ Girard's theorem ~~with the students~~ and ~~then~~ ~~we discussed~~ the basic features of the Einstein field equation. ~~The discussion of the~~ Einstein field equation is ~~already~~ proposed in high-school textbooks, but ~~we think that is not satisfactory.~~ In this work, we ~~propose~~ activities to enlighten the connection between the theory and actual physical models, like the simulation of Mercury motions around the Sun.

~~More works has to~~ be done on the presentation ~~of the theory~~ of general relativity if ~~we want~~ to present it in ~~a non rigorous, but yet correct,~~ way. We think that a great ~~aid~~ may come from numerical ~~methods~~. ~~It is true that numerical analysis and optimization theory are complicated and we suggest [20] for a more detailed introduction to them focused on didactic purposes.~~ ~~Yet~~ ~~their~~ implementation is simple and ~~powerful. They~~ can be used to simulate classical, relativistic and quantistic systems~~, and nowadays instrument~~ ~~like~~

electronics spreadsheet and high level programming language allow a simple implementation of the algorithm that can be done by students. Moreover, in nowadays society electronics, with products like *Arduino* and *Raspberry Pi*, and coding, with programming languages like *Python* and *Java* have a great grip on young students for the pratical application that they allow. We hope that, in the near future, the comprehension of a physical phenomena, its mathematical model and of technological implementations may integrate to form a homogeneous educational path.

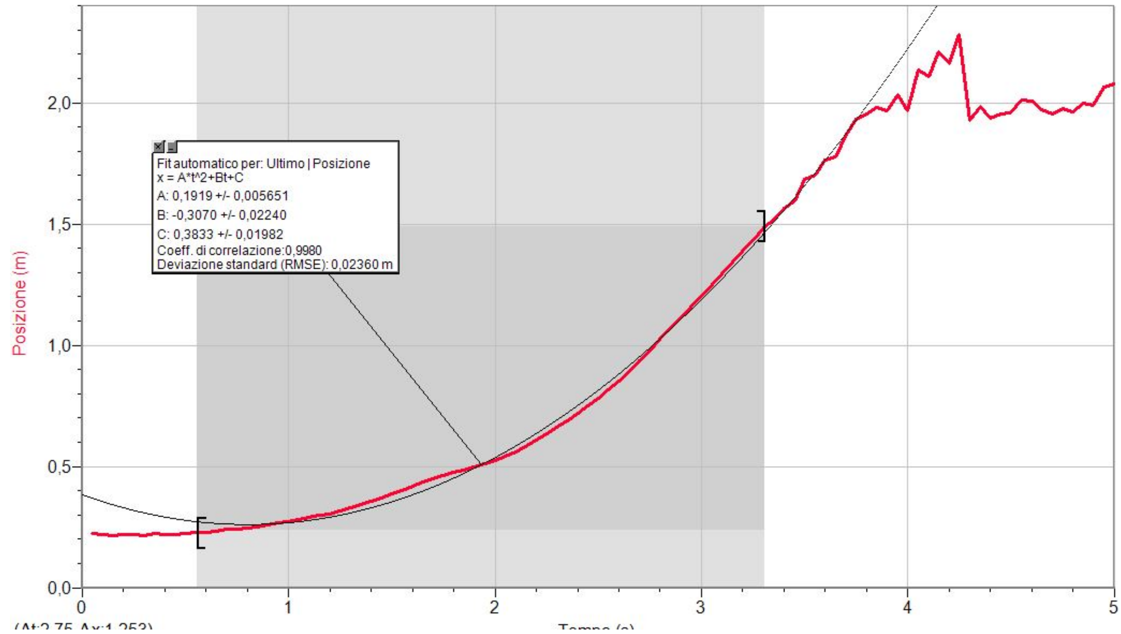
It is desirable that in future works more activities are dedicated to a single classroom, with no more than twenty students. This would allow more experiments to be performed by the students. It is often argued that an experimental approach is too slow for the time schedule of schools. From the experiences shared in the various meetings with the teachers the opposite emerged. This approach is slow in the beginning. Once students masters basic concepts acquired in laboratorial activities, teachers found a speed up in their lessons. We hope that in near future, more teachers will try didactics methodologies focused on experimentation performed by the students.

Appendix A

Students commentary to the space-time and velocity time diagrams

On Wednesday, May 9th I have gone to the "Liceo Scientifico Statale Carlo Urbani" hosted by professors *Margherita D'Urzo* and *Ilaria Limoncelli*. As described in chapter one, I proposed two hours of laboratorial activity to introduce classical mechanics from the Galilean transformation. As stated by the professors, one of the main merits of this lessons was that it connected many topics generally not seen as one. As *Ilaria Limoncelli* said, this has been a great recapitulation of the work done during the year, and it was done at the beginning of the year it could have been served as a good introduction for the subsequent work.
To test the efficacy of my seminary, I gave (under the suggestion of my supervisor) to the students the assignment to describe the graphs we produced in a relation.
We think that this is a very important activity, very often students did not know how to express their thoughts because they are not trained in describe something with written words. In the following, some of the work produced by the students is proposed.

Mercoledì 9 Maggio 2018, i ragazzi della classe III sez. H e G hanno seguito una lezione della durata di due ore con un docente universitario. Nella parte iniziale della lezione sono stati svolti vari esperimenti dai ragazzi stessi per studiare il moto rettilineo uniforme e uniformemente accelerato. L'esperimento seguente è stato svolto utilizzando un sensore di movimento e l'aiuto di un ragazzo che ha effettuato degli spostamenti davanti al sensore. Il grafico è il seguente:

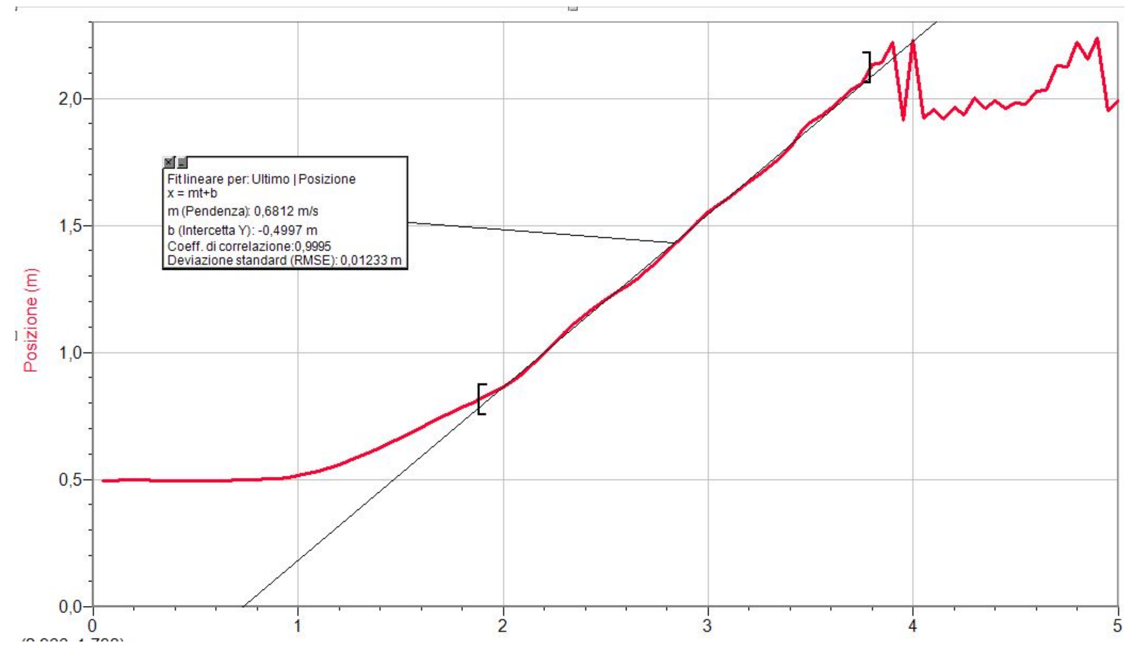


Durante l'esperimento il ragazzo ha compiuto uno spostamento retrogrado rispetto al sensore. Sull'asse delle ordinate è stato collocato lo spazio (m) percorso dall'alunno nell'intervallo di tempo (s) riportato sull'asse delle ascisse. Il movimento parte con velocità bassa, a una distanza di 0,8m , ma durante l'intero esperimento la velocità aumenta in maniera approssimativamente costante e il moto è assimilabile ad una parabola ed è un esempio di moto rettilineo uniformemente accelerato. Nell' ultimo secondo di movimento durante la misurazione, il sensore ha rilevato la posizione di altri oggetti poiché l'alunno si trovava al di fuori del raggio d'azione del sensore.

SALVATORE ARMELLINO

Il giorno09/05/2018 le classi 3H e 3G si sono recate presso il laboratorio di fisica dell'Istituto Filippo Silvestri per eseguire una lezione in compagnia di un docente universitario, in modo tale da approfondire gli studi sostenuti in precedenza. L'obiettivo era quello di dimostrare le nozioni acquisite, in particolar modo il moto dei corpi. Dalle seguenti sperimentazioni sono emersi vari grafici:

Per conseguire il nostro fine abbiamo adoperato:
Un sensore di movimento;
Un corpo.

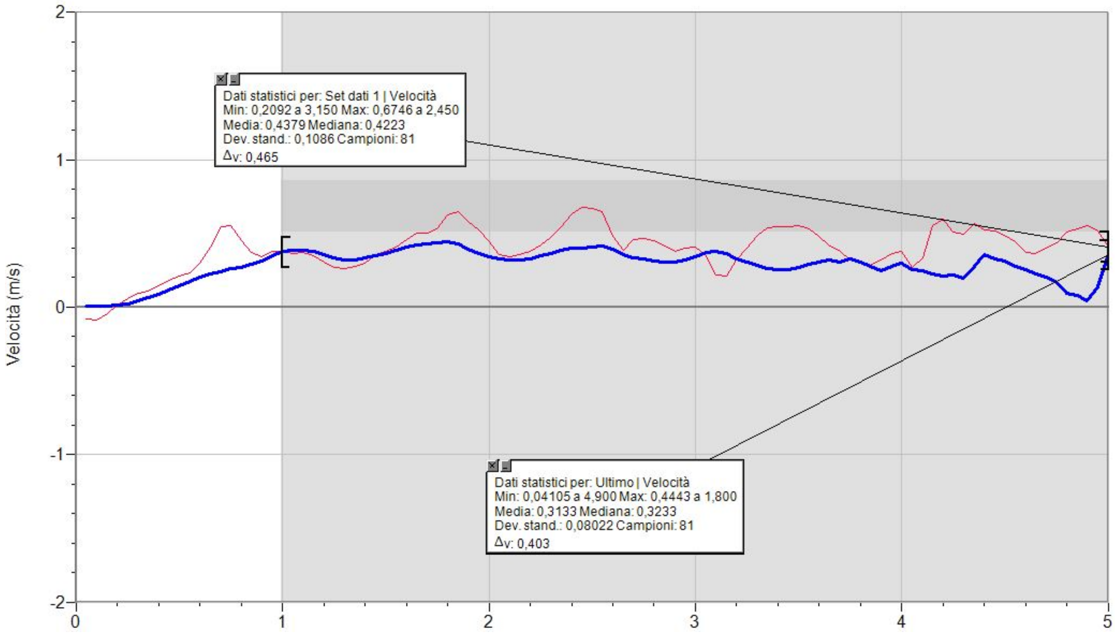


Durante questo esperimento un ragazzo/a si è spostato davanti al sensore, che come ben sappiamo è uno strumento capace di misurare i raggi infrarossi irradiati da un corpo nel suo campo di vista. Nel seguente grafico cartesiano, sull'asse delle ordinate è collocato lo spazio (m) percorso dal corpo mentre sull'asse delle ascisse, l'intervallo di tempo impiegato. Il corpo per il primo secondo è in stasi, posto a 0,5 m dal sensore, dopo un iniziale spostamentoil corpo mantiene una velocità approssimativamente costante, anche se sono presenti alcune imprecisioni dovute ai passi. Dal quarto secondo in poi il corpo è uscito dal campo visivo del sensore, esso quindi ha rilevato la posizione di altri oggetti e ne conseguela variazione improvvisa del moto.

EMANUELE COZZOLINO

Durante il corso del secondo quadrimestre dell'anno 2018, il 9 maggio, le classi terze, in particolare le sezioni G e H, sono state invitate a seguire una lezione di approfondimento sul moto di un corpo, studiato nel biennio. Durante la lezione un alunno universitario ha mostrato come attraverso un sensore di movimento, un apparecchio che rileva spostamento, velocità e accelerazione di un corpo entro un raggio visivo, si possono realizzare e analizzare grafici sul moto rettilineo uniforme e uniformemente accelerato. Il responsabile ha invitato dei ragazzi, volontari, a testare di persona l'esperimento, muovendosi davanti al sensore. Tra i grafici realizzati noi analizzeremo questo.

Questo grafico sovrappone due accelerazioni di 2 corpi, il quale uno parte con una velocità (v) uguale a 0 (indicato con la linea blu) e l'altro con una pari a circa -0.1 m/s



(indicato con la linea rossa).

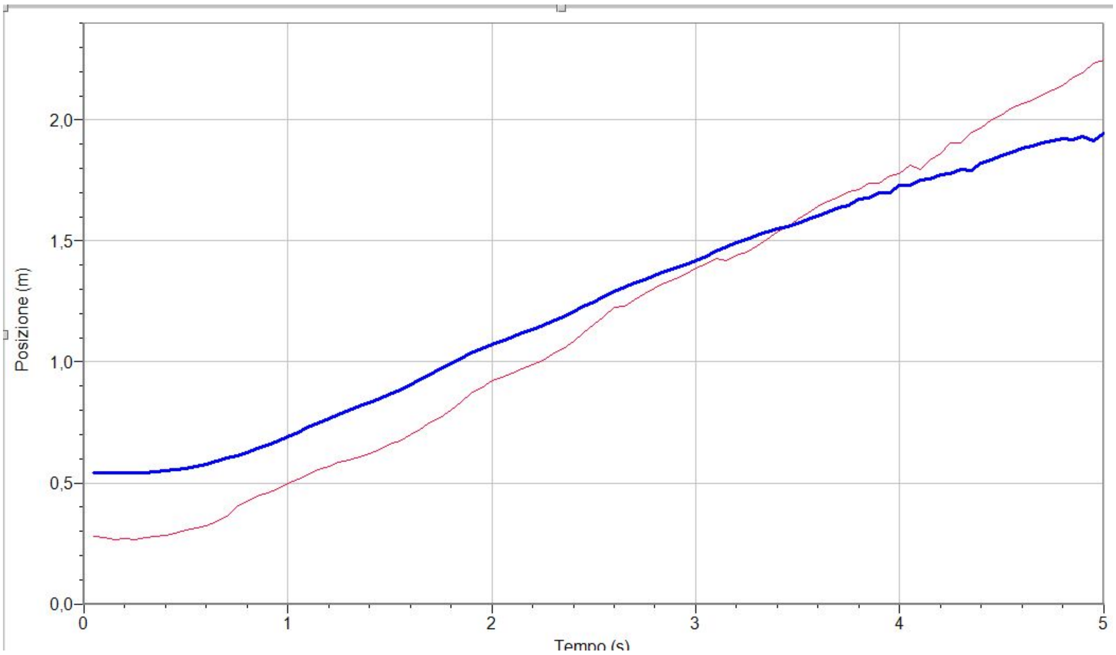
Il corpo A (blu) mantiene più o meno invariata la sua velocità, anche se con alcune imprecisioni dovute all'accelerazione non nulla del volontario, fino al secondo 4.3, dove troviamo un aumento di velocità, durato intorno ai 0.2 s, seguito da una decelerazione, fino a un decimo di secondo prima della fine dell'esperimento, per concludersi con un brusco aumento della velocità, da 0.02 m/s a 0.4m/s nell'arco di 0.1s, quindi con accelerazione (a) pari a $3.8m/s^2$.

Il corpo B (rosso) ha un moto, accelerato e decelerato, meno uniforme di quello A. Infatti parte con una velocità negativa, come scritto in precedenza, per poi essere sempre positiva con salti continui che indicano l'aumento e la diminuzione della velocità, che noi possiamo notare nel grafico ma che nel quotidiano si possono non notare per le variazioni molto piccole di velocità in un piccolo arco di tempo.

GIUSEPPE FIENGO

Mercoledì 9 Maggio, i ragazzi delle classi III H e III G hanno seguito la lezione di uno studente universitario, presso il laboratorio di fisica del liceo scientifico Filippo Silvestri. Il fine di questa lezione era di approfondire gli argomenti studiati durane l'anno, precisamente il moto dei corpi, medianteun sensore di movimento. Si sono ottenuti diversi grafici, uno di questi è il seguente:

Questo grafico è stato realizzato grazie a due alunni, Luca e Sara. Durante l'esperimento la prima ad essersi mossa è stata Sara, la quale si è allontanata dal sensore, partendo



da una posizione iniziale approssimativamente pari a 0,3m. Successivamente è stato sovrapposto il grafico posizione/tempo ottenuto dall'alunno Luca, il quale si è spostato da una posizione approssimativa di 0,5m, procedendo allo stesso modo di Sara. Entrambi i grafici possono essere paragonati ad una retta, di conseguenza entrambi gli alunni si muovevano di moto rettilineo uniforme, nonostante delle piccole oscillazioni dovute ai passi.

Relazione di: *Elisabetta Naldi* III H

Appendix B

Python and Jupyter Notebook

As may be found on *https://www.python.org/*, *python* is a great object-oriented, *interpreted, and interactive programming language. It is often compared (favorably of course :-)) to Lisp, Tcl, Perl, Ruby, C#, Visual Basic, Visual Fox Pro, Scheme or Java... and it's much more fun.*
Python combines remarkable power with very clear syntax. It has modules, classes, exceptions, very high level dynamic data types, and dynamic typing. There are interfaces to many system calls and libraries, as well as to various windowing systems. New built-in modules are easily written in C or C++ (or other languages, depending on the chosen implementation). Python is also usable as an extension language for applications written in other languages that need easy-to-use scripting or automation interfaces.
Python can be downloaded from the aforementioned website, and in this work it is used together with the jupyter notebook. With this tools students can build, in a simple and intuitive way, interactive graphs and simulations of physical systems. Indeed the numerical approach allows to study systems that could not be solved with analytic methods both because it is not possible or because students have not the tools. Solution of the equation of motion with the Euler methods are a valid example, and python has well tested libraries for visualizing data and create interactive graphs.
Teachers should not be afraid in requiring coding skill from their students, projects like *Coding in your Classroom, Now!* as described on the web page
https://platform.europeanmoocs.eu/course_coding_in_your_classroom_now
introduce programming already in elementary schools (also in Italy).
Hopefully, in some years the coding abilities will be considered at the same level of algebraic and geometric ones, since they form a semantic register powerful and intuitive in which thinking and solving problems.
A powerful tools for programming in python (or other programming languages) is the *jupyter notebook.*

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A powerful tools for programming in python (or other programming languages) is the *jupyter notebook*

"But it is true that if we look at a glass of wine closely enough we see the entire universe. There are the things of physics: the twisting liquid which evaporates depending on the wind and weather, the reflection in the glass; and our imagination adds atoms. The glass is a distillation of the earth's rocks, and in its composition we see the secrets of the universe's age, and the evolution of stars. What strange array of chemicals are in the wine? How did they come to be? There are the ferments, the enzymes, the substrates, and the products. There in wine is found the great generalization; all life is fermentation. Nobody can discover the chemistry of wine without discovering, as did Louis Pasteur, the cause of much disease. How vivid is the claret, pressing its existence into the consciousness that watches it! If our small minds, for some convenience, divide this glass of wine, this universe, into parts – physics, biology, geology, astronomy, psychology, and so on – remember that nature does not know it! So let us put it all back together, not forgetting ultimately what it is for. Let it give us one more final pleasure; drink it and forget it all! - *The Feynman lectures on physics*"

Richard Philips Feynmann - The Feynmann lectures on physics

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